8. A hot plate (1 m × 1 m), at 150°C is to be cooled by attaching on its surface, 10,000 number of cylindrical, pin fins of each, 3 mm diameter and 3 cm long. Surrounding air is at 25°C. Heat transfer coefficient between the fin surfaces and the surroundings is 30 W/(m^{2}C). Determine:
   (i) overall surface effectiveness
   (ii) heat transfer rate, with the fins in place
   (iii) heat transfer rate from the plate, if there were no fins
   (iv) decrease in thermal resistance due to attaching the fins.
9. An aluminium fins are fixed on one side (size: 1 m x 1 m), of an electronic device to increase the heat dissipation. Fins are of rectangular cross section, 0.2 cm thick and 3 cm long. There are 100 fins per metre. Convection heat transfer coefficient for both the plate and the fins is 30 W/(m^{2}K). Determine the percentage increase in the rate of heat transfer due to attaching the fins.
10. An iron bar, 15 mm in diameter, spans the distance between two plates, 50 cm apart. Air at 25°C flows in the space between the plates resulting in heat transfer coefficient of 15 W/(m^{2}K). Calculate the heat transfer and temperature at the middle of the bar, if the plates are maintained at 125°C each. For iron, k = 45 W/(mK).
11. Two ends of a 6 mm diameter copper rod (U–shaped) having k = 330 W/(mK), are rigidly connected to a vertical wall as shown in Fig. Problem 6.11. Wall temperature is constant at 100°C. Developed length of the rod is 50 cm and is exposed to air at 30°C. Combined convective and radiative heat transfer coefficient is 30 W/(m^2 K). Calculate:
   (i) the temperature at the centre of the rod
   (ii) net heat transfer from the rod to air.

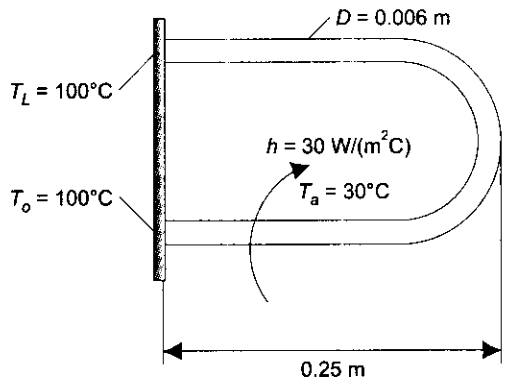

FIGURE Problem 6.11 U-shaped rod, both ends fixed to a wall

12. A steel rod (k = 55 W/(mK)), of length 50 cm, diameter 2.5 cm, has its two ends maintained at 150°C and 60°C. Ambient air, to which heat is dissipated by the rod, is at 25°C and the heat transfer coefficient is 20 W/(m^2 K). Determine:
   (i) minimum temperature in the rod
   (ii) temperature at the mid-point of the rod, and
   (iii) heat transfer rates from the left and right ends.
13. A Hg-thermometer placed in a well filled with oil, is required to measure the temperature of compressed air flowing in a pipe. The well is 14 cm long and is made of steel 1.5 mm thick. The temperature indicated by the thermometer is 100°C. The pipe wall temperature is 50°C. The film coefficient outside the wall is 30W/(m^{2}C). Estimate the % error in measurement of temperature of air. k for steel = 40W/(mC).

CHAPTER

7

Transient Heat Conduction

7.1 Introduction

In chapter 3, we derived the general differential equation for conduction and then applied it to problems of increasing complexity, e.g. first, we studied heat transfer in simple geometries without heat generation and then we studied heat transfer when there was internal heat generation. In all these problems, steady state heat transfer was assumed, i.e. the temperature within the solid was only a function of position and did not depend on time, i.e. mathematically, $T = T(x, y, z)$. However, all the process equipments used in engineering practice, such as boilers, heat exchangers, regenerators, etc. have to pass through an unsteady state in the beginning when the process is started, and, they reach a steady state after sufficient time has elapsed. Or, as another example, a billet being quenched in an oil bath, goes through temperature variations with both position and time before it attains a steady state. Conduction heat transfer in such an unsteady state is known as transient heat conduction or, unsteady state conduction, or time dependent conduction. Obviously, in transient conduction, temperature depends not only on position in the solid, but also on time. So, mathematically, this can be written as $T = T(x, y, z, \tau)$, where τ represents the time coordinate.

Naturally, solutions for transient conduction problems are a little more complicated compared to steady state analysis, since now, an additional parameter, namely time (τ) is involved.

Typical examples of transient conduction occur in:

(i) heat exchangers
(ii) boiler tubes
(iii) cooling of cylinder heads in I.C. engines
(iv) heat treatment of engineering components and quenching of ingots
(v) heating of electric irons
(vi) heating and cooling of buildings
(vii) freezing of foods, etc.

Two types of transient conduction may be identified:

(a) periodic heat flow problems, where the temperatures vary on a regular, periodic basis, e.g. in I.C. engine cylinders, alternate heating and cooling of earth during a 24 hr cycle (by sun) etc.
(b) non-periodic heat flow problems, where temperature varies in a non-linear manner with time.

To solve a given one-dimensional, transient conduction problem, one could start with one of the relevant general differential equations discussed in chapter 3 and by solving it in conjunction with appropriate boundary conditions, and get the temperature distribution as a function of position and time. For example, for one-dimensional conduction, in Cartesian coordinates, we have:

$$\frac{d^2T}{dx^2} = \frac{1}{\alpha}\cdot\frac{dT}{d\tau}$$ *-without heat generation)*

and, $$\frac{d^2T}{dx^2} + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{dT}{d\tau}$$ (*with heat generation.*)

However, there is a set of problems encountered in practice, where the temperature gradients within the solid are very small, (i.e. the internal resistance to conduction is negligible) which can be solved simply by applying the energy balance principle. Consider for example, a small body made of, say, copper, at a high temperature, being quenched in a medium like oil. Then, the body loses heat to the medium. Heat flows by conduction from within the body to the surface and then, by convection to the medium. When the body is very small or when the thermal conductivity of the material of the body is very large, temperature gradients within the body will be very small and may be neglected. In such a case, temperature within the body is only a function of time and is independent of spatial coordinates, i.e. the whole body acts as lump and temperatures of all points within the body decrease (or increase if the object is being heated) uniformly en-mass. Heat transfer process from the body, in this case, is controlled by the convection resistance at the surface rather than by the conduction resistance in the solid. Such an analysis, where the internal resistance of the body for heat conduction is negligible and the whole body may be treated as a lump as far as temperature increase or decrease is concerned, is known as **lumped system analysis**.

In this chapter, first, we shall study the lumped system analysis; then, we shall present analytical and chart solutions for some of the practically important transient conduction problems for the cases of a large slab, long cylinder, sphere and a semi-inifinite medium. Finally, product solution method of solving multidimensional transient conduction problems will be explained.

7.2 Lumped System Analysis (Newtonian Heating or Cooling)

As mentioned above, in lumped system analysis, the internal conduction resistance of the body to heat flow (i.e. $L/(k.A)$) is negligible compared to the convective resistance (i.e. $1/(h.A)$) at the surface. So, the temperature of the body, no doubt, varies with time, but at any given instant, the temperature within the body is uniform and is independent of position, i.e. $T = T(\tau)$ only. Practical examples of such cases are: heat treatment of small metal pieces, measurement of temperature with a thermocouple or thermometer, etc., where the internal resistance of the object for heat conduction may be considered as negligible.

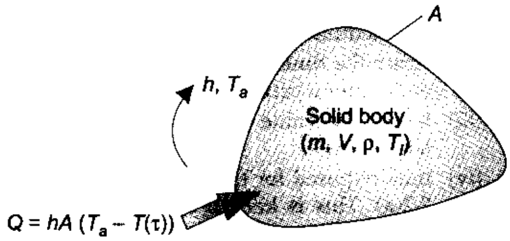

FIGURE 7.1 Lumped system analysis

Analysis:

Consider a solid body of arbitrary shape, volume V, mass m, density ρ, surface area A, and specific heat C_p. See Fig. 7.1. To start with, at $\tau = 0$, let the temperature throughout the body be uniform at $T = T_i$. At the instant $\tau = 0$, let the body be suddenly placed in a medium at a temperature of T_a, as shown. For the sake of analysis, let us assume that $T_a > T_i$; however, same analysis is valid for $T_a < T_i$ too. Then, heat will be transferred from the medium to the body and the temperature of the body will increase with time. Let the temperature of the body rise by a differential amount dT in a differential time interval $d\tau$, thus increasing the internal energy of the solid.

Writing an energy balance for this situation:

Amount of heat transferred into the body in time interval $d\tau$ =

Increase in the internal energy of the body in time interval $d\tau$

i.e. $$h \cdot A \cdot (T_a - T(\tau)) \cdot d\tau = m \cdot C_p \cdot dT = \rho \cdot Cp \cdot V \cdot dT \quad ...(7.1)$$

since $$m = \rho \cdot V$$

Now, since T_a is a constant, we can write:

$$dT = d(T(\tau) - T_a)$$

Therefore,

$$\frac{d(T(\tau) - T_a)}{T(\tau) - T_a} = \frac{-h \cdot A}{\rho \cdot C_p \cdot V} \cdot d\tau \quad ...(7.2)$$

Integrating between $\tau = 0$ (i.e. $T = T_i$) and any τ, (i.e. $T = T(\tau)$),

$$\ln\left(\frac{(T(\tau) - T_a)}{T_i - T_a}\right) = \frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}$$

i.e.

$$\frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \qquad \text{...(7.3)}$$

Now, let:

$$\frac{\rho \cdot C_p \cdot V}{h \cdot A} = t$$

where, t is known as thermal time constant and has units of time.

Therefore, Eq. 7.3 is written as:

$$\frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-\tau}{t}\right) \qquad \text{...(7.4)}$$

Now denoting $\theta = (T(\tau) - T_a)$, we write Eq. 7.4 compactly as:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-\tau}{t}\right) \qquad \text{...(7.5)}$$

Eq. 7.5 gives the temperature distribution in a solid as a function of time, when the internal resistance of the solid for conduction is negligible compared to the convective resistance at its surface.

Eq. 7.5 is represented graphically in Fig. 7.2.

From Eq. 7.5 and Fig 7.2, we note:

(i) temperature distribution is exponential, i.e. temperature changes rapidly initially and approaches that of the medium exponentially.

(ii) either the time required by the body to reach a certain temperature or the temperature attained by the body after a certain time interval, can be found out from Eq. 7.5.

(iii) larger the value of time constant t, longer is the time required for the body to reach a particular temperature.

(iv) time required for the body to attain 36.8% of the applied temperature difference is indicated in the Fig. 7.2(a). This is known as one time period and is of importance in connection with measurement of temperatures with thermocouples. Larger the value of time constant, larger is the time period. We shall comment on this later in this chapter.

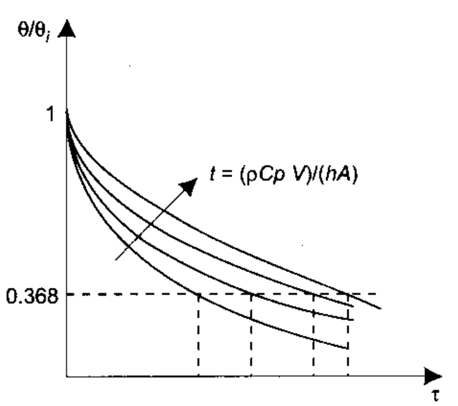

FIGURE 7.2(a) Temperature variation with time in a lumped system

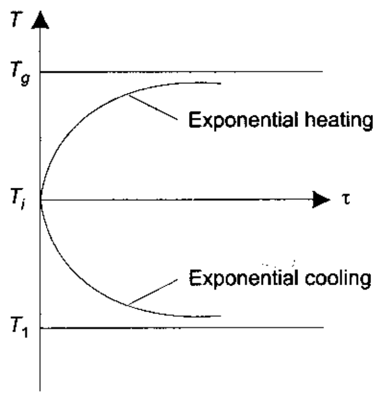

FIGURE 7.2(b) Newtonian heating and cooling

Instantaneous heat transfer:

At any instant τ, heat transfer between the body and the environment is easily calculated since we have the temperature distribution from Eq. 7.4:

$$Q(\tau) = m \cdot C_p \cdot \frac{dT(\tau)}{d\tau}, \text{ W} \quad \text{...(7.6a)}$$

At that instant, heat transfer must also be equal to:

$$Q(\tau) = h \cdot A \cdot (T(\tau) - T_a), \text{ W} \quad \text{...(7.6b)}$$

Total heat transfer:

Total heat transferred during $\tau = 0$ to $\tau = \tau$, is equal to the change in internal energy of the body:

$$Q_{tot} = m \cdot C_p \cdot (T(\tau) - T_i), \text{ J} \quad \text{...(7.7a)}$$

Q_{tot} may also be calculated by integrating Eq. 7.6a:

$$Q_{tot} = \int_0^{\tau} Q(\tau) d\tau, \text{ J} \quad \text{...(7.7b)}$$

Maximum heat transferred:

When the body reaches the temperature of the environment, obviously, maximum heat has been transferred:

$$Q_{max} = m \cdot C_p \cdot (T_a - T_i), \text{ J} \quad \text{...(7.8)}$$

If Q_{max} is negative, it means that the body has lost heat, and if Q_{max} is positive, then body has gained heat.

7.3 Criteria for Lumped System Analysis (Biot Number and Fourier Number)

For the simple analysis made above, we had the fundamental assumption that the internal conductive resistance of the body was negligible as compared to the convective resistance at its surface. This was stated in a rather qualitative way. Now, let us study the criteria required for the lumped system analysis to be applicable.

Consider a plane slab in steady state, transferring heat to a fluid on its surface with a heat transfer coefficient of h, as shown in Fig. 7.3. (The criterion arrived at is readily extended to transient conditions later.)

Let the surface on the left be maintained at temperature T_1 and the surface on the right is at a temperature of T_2 as a result of heat being lost to a fluid at temperature T_a, flowing with a heat transfer coefficient h. Writing an energy balance at the right hand surface,

$$\frac{k \cdot A}{L} \cdot (T_1 - T_2) = h \cdot A \cdot (T_2 - T_a)$$

Rearranging,

$$\frac{T_1 - T_2}{T_2 - T_a} = \frac{\left(\frac{L}{k \cdot A}\right)}{\left(\frac{1}{h \cdot A}\right)} = \frac{R_{cond}}{R_{conv}} = \frac{h \cdot L}{k} = Bi \quad \text{...(7.9)}$$

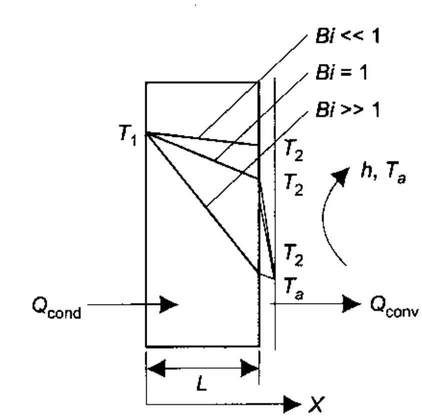

FIGURE 7.3(a) Biot number and temperature distribution in a plane wall

The term, $(h.L)/k$, appearing on the RHS of Eq. 7.9 is a dimensionless number, known as **Biot number**.

Biot number is a measure of the temperature drop in the solid relative to the temperature drop in the convective layer. It is also interpreted as the ratio of conductive resistance in the solid to the convective resistance at its surface. This is precisely the criterion we are looking for. Note from Fig. 7.3(a) the temperature profile for $Bi \ll 1$. It suggests that one can assume a uniform temperature distribution within the solid if $Bi \ll 1$.

Situation during transient conduction is shown in Fig. 7.3(b). It may be observed that temperature distribution is a strong function of Biot number. For $Bi \ll 1$, temperature gradient in the solid is small and temperature can be taken as a function of time only. Note also that for $Bi \gg 1$, temperature drop across the solid is much larger than that across the convective layer at the surface.

Therefore, to fix the criterion for which lumped system analysis is applicable, let us define Biot number, in general, as follows:

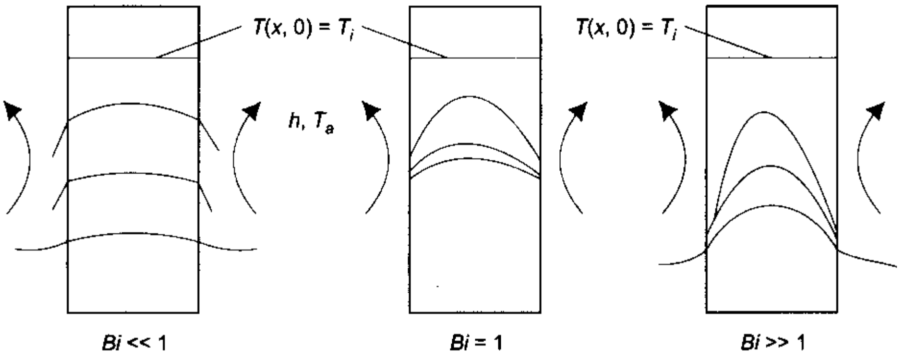

FIGURE 7.3(b) Biot number and transient temperature distribution in a plane wall

$$Bi = \frac{h \cdot L_c}{k} \qquad \text{...(7.10)}$$

where, h is the heat transfer coefficient between the solid surface and the surroundings, k is the thermal conductivity of the solid, and L_c is a characteristic length defined as the ratio of the volume of the body to its surface area, i.e.

$$L_c = \frac{V}{A}$$

With this definition of Bi and L_c, for solids such as a plane slab, long cylinder and sphere, it is found that transient temperature distribution within the solid at any instant is uniform, with the error being less than about 5%, if the following criterion is satisfied:

$$Bi = \frac{h \cdot L_c}{k} < 0.1 \qquad \text{...(7.11)}$$

In other words, if the conduction resistance of the body is less than 10% of the convective resistance at its surface, the temperature distribution within the body will be uniform within an error of 5%, during transient conditions.

L_c for common shapes:

(i) Plane wall (thickness $2L$): $L_c = \frac{A \cdot 2 \cdot L}{2 \cdot A} = L$ = half-thickness of wall

(ii) Long cylinder, radius R: $L_c = \frac{\pi \cdot R^2 \cdot L}{2 \cdot \pi \cdot R \cdot L} = \frac{R}{2}$

(iii) Sphere, radius, R: $L_c = \frac{\frac{4}{3} \cdot \pi \cdot R^3}{4 \cdot \pi \cdot R^2} = \frac{R}{3}$

(iv) Cube, side L: $L_c = \frac{L^3}{6 \cdot L^2} = \frac{L}{6}$

Therefore, we can write Eq. 7.3 as:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \quad \text{if } Bi < 0.1 \qquad \text{...(7.12)}$$

Eq. 7.12 is important. Its appliation to a given problem is very simple and solution of any transient conduction problem must begin with examining if the criterion, $Bi < 0.1$ is satisfied to see if Eq. 7.12 could be applied.

Now, the term $(h \cdot A \cdot \tau)/(\rho \cdot C_p \cdot V)$ can be written as follows:

$$\frac{h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V} = \left(\frac{h \cdot L_c}{k}\right) \cdot \left(\frac{k \cdot \tau}{\rho \cdot C_p \cdot L_c^2}\right) = \left(\frac{h \cdot L_c}{k}\right) \cdot \left(\frac{\alpha \cdot \tau}{L_c^2}\right) = Bi \cdot Fo \qquad \text{...since } \frac{V}{A} = L_c$$

where, $Fo = \frac{\alpha \cdot \tau}{L_c^2}$ = **Fourier number, or relative time.**

Fourier number, like Biot number, is an important parameter in transient heat transfer problems. It is also known as 'dimensionless time'. Fourier number signifies the degree of penetration of heating or cooling effect through a solid. For small Fo, large τ will be required to get significant temperature changes.

With the aforesaid definitions of Biot number and Fourier number, now, we can rewrite Eq. 7.12 as:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp(-Bi \cdot Fo) \quad \text{if } Bi < 0.1 \qquad \text{...(7.13)}$$

Eq. 7.13 is plotted in Fig. 7.4 below. On the X-axis, $(Bi.Fo)$ is plotted against θ/θ_i on Y-axis. As expected, the graph is a straight line, with a negative slope when the Y-axis has logarithmic scale. Remember that this graph is for the cases where lumped system analysis is applicable, i.e. $Bi < 0.1$.

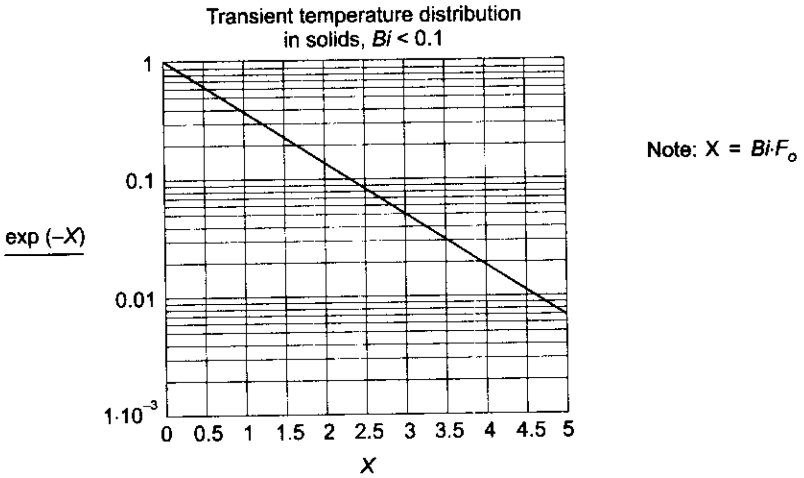

FIGURE 7.4 Dimensionless temperature distribution in solids during transient heat transfer, $(Bi < 0.1)$, for lumped system analysis

7.4 Response Time of a Thermocouple

Lumped system analysis is usefully applied in the case of temperature measurement with a thermometer or a thermocouple. Obviously, it is desirable that the thermocouple indicates the source temperature as fast as possible. If the thermocouple is measuring changing temperatures, then also, it should follow the temperature changes at a rate faster than the rate of temperature change. 'Response time' of a thermocouple is defined as the time taken by it to reach the source temperature.

Consider Eq. 7.12:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \quad \text{if } Bi < 0.1 \qquad \text{...(7.12)}$$

For rapid response, the term $(hA\tau)/(\rho C_p V)$ should be large so that the exponential term will reach zero faster. This means that:

(i) increase (A/V), i.e. decrease the wire diameter

(ii) decrease density and specific heat, and

(iii) increase the value of heat transfer coefficient h.

As mentioned earlier, the quantity $(\rho C_p V)/(h \cdot A)$ is known as 'thermal time constant', t, of the measuring system and has units of time. At $\tau = t$, i.e. at a time interval of one time constant, we have:

$$\frac{T(\tau) - T_a}{T_i - T_a} = e^{-1} = 0.368 \qquad ...(7.14)$$

From Eq. 7.14, it is clear that after an interval of time equal to one time constant of the given temperature measuring system, the temperature difference between the body (thermocouple) and the source would be 36.8% of the initial temperature difference, i.e. the temperature difference would be reduced by 63.2%.

Time required by a thermocouple to attain 63.2% of the value of initial temperature difference is called its **sensitivity**.

For good response, obviously, the response time of thermocouple should be low. As a thumb rule, it is recommended that while using a thermocouple to measure temperatures, reading of the thermocouple should be taken after a time equal to about four time periods has elapsed.

Example 7.1. A steel ball of 5 cm diameter initially at a uniform temperature of 450°C is suddenly placed in an environment at 100°C. Heat transfer coefficient h, between the steel ball and the fluid is 10 W/(m^{2}K). For steel, c_p = 0.46 kJ/(kgK), ρ = 7800 kg/m^3, k = 35 W/(mK). Calculate the time required for the ball to reach a temperature of 150°C. Also, find the rate of cooling after 1 hr. Show graphically how the temperature of the sphere falls with time. [M.U.]

Solution.

Data:

$R := 25 \times 10^{-3}$ m $\quad \rho := 7800$ kg/m^3 $\quad C_p := 460$ J/(kgK) $\quad k := 35$ W/(mK) $\quad T_i := 450$°C $\quad T_a := 100$°C

$h := 10$ W/(m^{2}K) $\quad T := 150$°C $\quad A := 4 \cdot \pi \cdot R^2$, m^2 $\quad V := \frac{4}{3} \cdot \pi \cdot R^3$, m^3

First, calculate the Biot number:

$$Bi = \frac{h \cdot L_c}{k} = \frac{h}{k} \cdot \left(\frac{V}{A}\right) = \frac{h}{k} \cdot \frac{\left[\frac{4}{3} \cdot (\pi) \cdot R^3\right]}{4 \cdot \pi \cdot R^2}$$

i.e. $\quad Bi := \frac{h}{k} \cdot \frac{R}{3}$ *(define Biot number)*

i.e. $\quad Bi = 2.381 \times 10^{-3}$ *(Biot number.)*

Since $Bi < 0.1$, lumped system analysis is applicable, and the temperature variation within the solid will be within an error of 5%. Applying Eq. 7.12, we get:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \text{ if } Bi < 0.1 \qquad ...(7.12)$$

i.e. $\quad \frac{T - T_a}{T_i - T_a} = \exp\left(\frac{-\tau}{t}\right)$ where, t is the time constant.

And, **time constant** is given by:

$$t = \frac{\rho \cdot c_p \cdot V}{A \cdot h} = \frac{\rho \cdot c_p}{h} \cdot \frac{R}{3}$$ *(since for sphere, $V/A = R/3$)*

i.e. $\quad t := \frac{\rho \cdot c_p}{h} \cdot \frac{R}{3}$ *(define time constant, t)*

i.e. $\quad t = 2990$ s *(time constant)*

Therefore, we write:

$$\frac{150 - 100}{450 - 100} = \exp\left(\frac{-\tau}{2990}\right)$$ where, τ is the time required to reach 150°C

i.e. $\quad \ln\left(\frac{50}{350}\right) = \frac{-\tau}{2990}$

or, $\tau := -2990\ \ln\left(\frac{50}{350}\right)$ s *(define τ, the time required to reach 150°C)*

i.e. $\tau = 5.818 \times 10^3$ s ***(time required to reach 150°C.)***

i.e. $\tau = 1.616$ hrs.

Rate of cooling after 1 hr.:

i.e. $\tau := 3600$ s

From Eq. 7.12, we have:

$$T(\tau) := \left[(T_i - T_a)\cdot \exp\left(\frac{-h\cdot A\cdot \tau}{\rho\cdot c_p\cdot V}\right) + T_a\right] \quad \textit{(define } T(\tau))$$

i.e.

$$\frac{dT}{d\tau} = (T_i - T_a)\cdot\left[\frac{-h\cdot A}{\rho\cdot V\cdot c_p}\right]\cdot \exp\left(\frac{-h\cdot A\cdot \tau}{\rho\cdot c_p\cdot V}\right)\ \text{C/s} \quad \textit{(rate of cooling)}$$

i.e.

$$\frac{d}{d\tau}T(\tau) = -0.035\ \text{C/s} \quad \textbf{\textit{(rate of cooling after 1 hr.)}}$$

negative sign indicates that as time increases, temperature falls.

Note that in Mathcad, there is no need to separately differentiate and substitute the values. All that is done in one step as shown above.

To sketch the fall in temperature of sphere with time:

Temperature as a function of time is given by Eq. 7.12:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h\cdot A\cdot \tau}{\rho\cdot c_p\cdot V}\right) \quad \text{if } Bi < 0.1 \qquad ...(7.12)$$

i.e.

$$T(\tau) := T_a + (T_i - T_a)\cdot \exp\left(\frac{-h\cdot A\cdot \tau}{\rho\cdot c_p\cdot V}\right) \qquad ...(A)$$

We will plot Eq. A against different times, τ:

We use Mathcad to draw the graph. First, define a range variable τ, varying from 0 to say, 4 hrs, with an increment of 0.1 hrs. Then, choose x–y graph from the graph palette, and fill up the place holders on the X-axis and Y-axis with τ and $T(\tau)$, respectively. Click anywhere outside the graph region and immediately the graph appears:

$\tau := 0, 0.1, \ldots, 4$ *(define a range variable, τ varying from zero to 4 hrs, with an increment of 0.1 hrs.)*

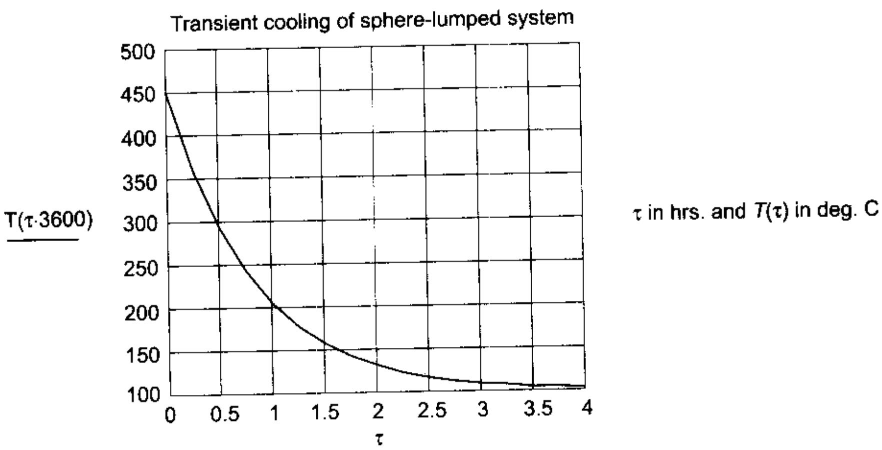

FIGURE Example 7.1 Transient cooling of a sphere considered as a lumped system

Note from the Fig. 7.4 how the cooling progresses with time. After about 4 hrs duration, the sphere approaches the temperature of the ambient. You can also verify from the graph that the time required for the sphere to reach 150°C is 1.616 hrs, as calculated earlier.

Example 7.2. A 50 cm x 50 cm copper slab, 6 mm thick, at a uniform temperature of 350°C, suddenly has its surface temperature lowered to 30°C. Find the time at which the slab temperature becomes 100°C. Given: $\rho = 9000$ kg/m^3, c_p = 0.38 kJ/(kgK), k = 370 W/(mK), h = 100 W/(m^{2}K). Also, find out the rate of cooling after 60 seconds.

Solution.

Data:

$L := 0.05$ m $\quad B := 0.05$ m *(breadth)* $\quad \delta := 0.006$ m *(thickness)* $\quad \rho := 9000$ kg/m^3 $\quad c_p := 380$ J/(kgK)

$k := 370$ W/(mK) $\quad T_i := 350°C \quad T_a := 30°C \quad h := 100$ W/(m^{2}K) $\quad T := 100°C$

$A := 2 \cdot L \cdot B$, m^2 $\quad V := L \cdot B \cdot \delta$, m^3

First, calculate the Biot number:

Characteristic length: $\quad L_c := \dfrac{V}{A}$ $\quad$ *(define characteristic length for the plate)*

i.e. $\quad L_c = 3 \times 10^{-3}$ m $\quad$ *(characteristic length for plate = half the thickness)*

Therefore, Biot number: $Bi := \dfrac{h \cdot L_c}{k}$ $\quad$ *(define Biot number)*

i.e. $\quad Bi = 8.108 \times 10^{-4}$ $\quad$ *(Biot number)*

Since $Bi < 0.1$, lumped system analysis is applicable, and the temperature variation within the solid will be within an error of 5%. Applying Eq. 7.12, we get:

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \quad \text{if } Bi < 0.1 \qquad ...(7.12)$$

i.e. $$\frac{T - T_a}{T_i - T_a} = \exp\left(\frac{-\tau}{t}\right) \quad \text{where, } t \text{ is the time constant.}$$

And, **time constant** is given by:

$$t = \frac{\rho \cdot c_p \cdot V}{A \cdot h} = \frac{\rho \cdot c_p}{h} \cdot L_c$$ *(since for plate, $V/A = L_c$)*

i.e. $\quad t := \dfrac{\rho \cdot c_p}{h} \cdot L_c$ $\quad$ *(define time constant, τ)*

i.e. $\quad t = 102.6$ s $\quad$ *(time constant)*

Therefore, we write:

$$\frac{100 - 30}{350 - 30} = \exp\left(\frac{-\tau}{102.6}\right) \quad \text{where, } \tau \text{ is the time required to reach 100°C}$$

i.e. $$\ln\left(\frac{70}{320}\right) = \frac{-\tau}{102.6}$$

or, $$\tau := -102.6 \ln\left(\frac{70}{320}\right) \text{ s}$$ *...define τ, the time required to reach 100°C*

i.e. $\quad \tau = 155.934$ s $\quad$ ***...time required to reach 100°C.***

i.e. $\quad \tau = 0.043$ hrs.

Rate of cooling after 60 s:

i.e. $\quad \tau = 60$ s

From Eq. 7.12, we have:

$$T(\tau) := \left[(T_i - T_a) \cdot \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot V \cdot c_p}\right) + T_a\right]$$ *...define $T(\tau)$*

i.e. $$\frac{dT}{d\tau} = (T_i - T_a) \cdot \left(\frac{h \cdot A}{\rho \cdot V \cdot c_p}\right) \cdot \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot V \cdot c_p}\right) \text{ C/s}$$ *...rate of cooling*

i.e. $$\frac{d}{d\tau} T(\tau) = -1.738 \text{ C/s}$$ ***...rate of cooling after 60 s.***

Negative sign indicates that as time increases, temperature falls.

Note again, that in Mathcad, there is no need to separately differentiate and substitute the values. All that is done in one step as shown above.

Example 7.3. A carbon steel (AISI 1010) shaft of 0.2 m diameter is heat treated in a gas-fired furnace whose gases are at 1200 K and provide a convection coefficient of 80 W/(m^{2}K). If the shaft enters the furnace at 300 K, how long must it remain in the furnace to achieve a centre line temperature of 900 K? Given thermophysical properties of AISI 1010 carbon steel: ρ = 7854 kg/m^3, k = 48.8 W/(mK), c_p = 559 J/(kgK).

Solution.

Data:

$R := 0.1$ m $\quad \rho := 7854$ kg/m^3 $\quad c_p := 559$ J/(kg K) $\quad k := 48.8$ W/(m K) $\quad T_i := 300$ K $\quad T_a := 1200$ K

$h := 80$ W/(m^{2}K) $\quad T := 900$ K $\quad A := 2\cdot\pi\cdot R\cdot L$, m^2 $\quad V := \pi\cdot R^2\cdot L$, m^3 $\quad L_c := \dfrac{V}{A}$

i.e. $\quad L_c = \dfrac{R}{2}$ m *(characteristic length)*

i.e. $\quad L_c = 0.05$ m *(characteristic length.)*

First, calculate the Biot number

$$Bi: = \frac{h\cdot L_c}{k}$$ *(define Biot number)*

i.e. $\quad Bi = 0.082$ *(Biot number)*

Since $Bi < 0.1$, lumped system analysis is applicable, and the temperature variation within the solid will be within an error of 5%. Applying Eq. 7.12, we get:

$$\frac{\theta}{\theta_i} = \frac{T(\tau)-T_a}{T_i-T_a} = \exp\left(\frac{-h\cdot A\cdot\tau}{\rho\cdot C_p\cdot V}\right) \quad \text{if } Bi < 0.1 \qquad ...(7.12)$$

i.e. $$\frac{T-T_a}{T_i-T_a} = \exp\left(\frac{-\tau}{t}\right)$$ where, t is the time constant.

And, **time constant** is given by:

$$t = \frac{\rho\cdot c_p\cdot V}{A\cdot h} = \frac{\rho\cdot c_p}{h}\cdot\frac{R}{2}$$ *(since for cylinder, $V/A = R/2$)*

i.e. $$t: = \frac{\rho\cdot c_p}{h}\cdot\frac{R}{2}$$ *(define time constant, t)*

i.e. $\quad t = 2.74399 \times 10^3$ s *(time constant)*

Therefore, we write:

$$\frac{900-1200}{300-1200} = \exp\left(\frac{-\tau}{2743.99}\right)$$ where, τ is the time required to reach 900 K

i.e. $$\ln\left(\frac{300}{900}\right) = \frac{-\tau}{2743.99}$$

or, $$\tau := -2743.99 \ln\left(\frac{300}{900}\right) \text{ s}$$ *(define τ, the time required to reach 900 K)*

i.e. $\quad \tau = 3.015 \times 10^3$ s ***(time required to reach 900 K.)***

i.e. $\quad \tau = 0.838$ hrs.

Example 7.4. A thermocouple (TC) junction is in the form of 8 mm sphere. Properties of the material are: c_p = 420 J/(kgK), ρ = 8000 kg/m^3, k = 40 W/(mK), and heat transfer coefficient, h = 45 W/(m^{2}K). Find, if the junction is initially at a temperature of 28°C and inserted in a stream of hot air at 300°C:

(i) the time constant of the TC

(ii) The TC is taken out from the hot air after 10 s and kept in still air at 30°C. Assuming 'h' in air as 10 W/(m^{2}K), find the temperature attained by the junction 15 s after removing from hot air stream. [M.U.]

Solution.

Data:

$R := 4 \times 10^{-3}$ m $\quad \rho := 8000$ kg/m^3 $\quad c_p := 420$ J(kg k) $\quad k := 40$ W/(mK) $\quad T_i := 28$°C $\quad T_a := 300$°C

$h := 45$ W/(m^{2}K) $\quad A := 4\cdot\pi\cdot R^2$, m^2 $\quad V := \dfrac{4}{3}\cdot\pi\cdot R^3$, m^3

First, calculate the Biot number:

$$Bi = \frac{h \cdot L_c}{k} = \frac{h}{k} \cdot \left(\frac{V}{A}\right) = \frac{h}{k} \cdot \frac{\left[\frac{4}{3} \cdot (\pi) \cdot R^3\right]}{4 \cdot \pi \cdot R^2}$$

i.e. $Bi := \frac{h}{k} \cdot \frac{R}{3}$ *(define Biot number)*

i.e. $Bi = 1.5 \times 10^{-3}$ *(Biot number)*

Since $Bi < 0.1$, lumped system analysis is applicable, and the temperature variation within the solid will be within an error of 5%.

See Fig. Example 7.4 (a).

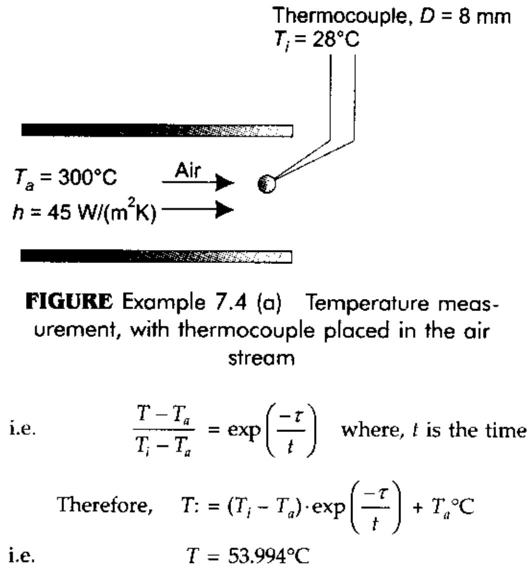

FIGURE Example 7.4 (a) Temperature measurement, with thermocouple placed in the air stream

Time constant is given by:

$$t = \frac{\rho \cdot c_p \cdot V}{A \cdot h} = \frac{\rho \cdot c_p}{h} \cdot \frac{R}{3}$$ *(since for sphere, $V/A = R/3$)*

i.e. $t := \frac{\rho \cdot c_p}{h} \cdot \frac{R}{3}$ *...define time constant, t)*

i.e. $t = 99.556$ s ***(time constant.)***

Temperature of TC after 10s:

$\tau := 10$ s (time duration for which TC is kept in the stream at 300°C)

We use Eq. 7.12, i.e.

$$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \text{ if } Bi < 0.1 \qquad \text{...(7.12)}$$

i.e. $\frac{T - T_a}{T_i - T_a} = \exp\left(\frac{-\tau}{t}\right)$ where, t is the time constant.

Therefore, $T := (T_i - T_a) \cdot \exp\left(\frac{-\tau}{t}\right) + T_a$°C *(define temperature of TC after 10 s in the stream)*

i.e. $T = 53.994$°C *(temperature of TC 10 s after it is placed in the stream at 30°C)*

(b) Now, this TC is removed from the stream at 300°C and placed in still air at 30°C. So, the temperature of 53.994°C becomes initial temperature T_i for this case:

i.e. new T_i: $T_i := 53.994$°C *(initial temperature when the TC is placed in still air)*

And, new τ: $\tau := 15$ s *(duration for which TC is kept in still air)*

And, new T_a: $T_a := 30$°C *(new temperature of ambient)*

And, new h: $h := 10$ W/(m^{2}K) *(heat transfer coefficient in still air.)*

See Fig. Example 7.4 (b).

And, new time constant:

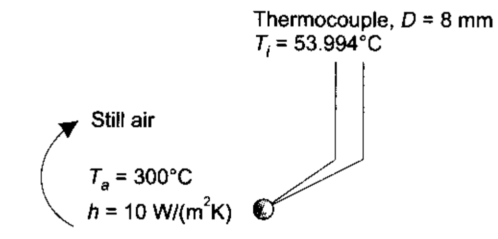

FIGURE Example 7.4 (b) Temperature measurement, with thermocouple placed in still air

i.e. $t := \frac{\rho \cdot c_p}{h} \cdot \frac{R}{3}$ *(define time constant, t)*

i.e. $t = 448$ s *(time constant)*

Therefore, $T := (T_i - T_a) \cdot \exp\left(\frac{-\tau}{t}\right) + T_a$ °C

...define temperature *of TC after 15 s in still air*

i.e. $T = 53.204$°C ***(temperature of TC 15 s after it is placed in still air at 30°C.***

7.5 Mixed Boundary Condition

In the cases studied so far, transient conduction was induced in the solid by subjecting it to convection on all its sides.

However, this need not be always so. Transient conditions can also be induced in a body by having radiation on any of its surfaces, or by subjecting any of its surfaces to electrical heating, or by internal heat generation caused by flow of electric current.

In the general case, where transient conditions are induced in a body by the combined effect of convection, radiation and heat generation, the controlling differential equation can be derived in the usual way, by writing an energy balance on the body taken as a control volume, i.e. net energy entering into the body results in an increase in the internal energy of the body. However, the resulting differential equation will be a nonlinear one, and is not amenable to exact analytic solution, and has to be solved by approximate finite difference methods.

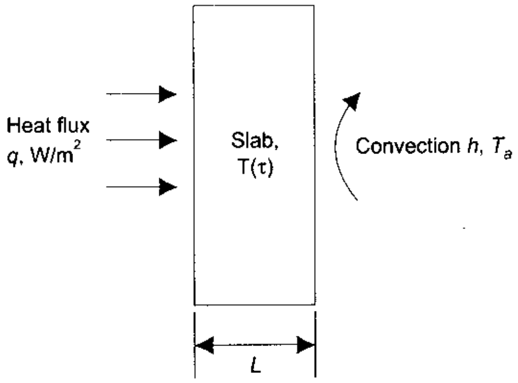

FIGURE 7.5 Transient conduction in a slab with mixed boundary conditions

Let us analyse one particular case (which is quite common), where one boundary surface is subjected to a uniform heat flux and the other boundary surface is subjected to convection. See Fig. 7.5.

As shown in the Fig. 7.5, a slab of thickness L, is subjected to a uniform heat flux $q\,(\mathrm{W/m^2})$ at its left face and loses heat by convection on its right face to a fluid at a temperature T_a, with a heat transfer coefficient, h. Then, applying energy balance for this case, we write:

(Energy going into the slab – Energy leaving the slab) = Increase in internal energy of the slab

i.e. $$q \cdot A - h \cdot A \cdot (T(\tau) - T_a) = \rho \cdot C_p \cdot A \cdot L \cdot \frac{dT(\tau)}{d\tau}$$

i.e. $$\frac{dT(\tau)}{d\tau} + \frac{h \cdot (T(\tau) - T_a)}{\rho \cdot C_p \cdot L} - \frac{q}{\rho \cdot C_p \cdot L} = 0 \qquad ...(7.15)$$

Substituting: $\theta = T(\tau) - T_a$ i.e. $\dfrac{d\theta}{d\tau} = \dfrac{dT(\tau)}{d\tau}$

and, $$a = \frac{h \cdot A}{\rho \cdot V \cdot C_p}$$

and, $$b = \frac{q \cdot A}{\rho \cdot V \cdot C_p} \quad \text{(remember: } A/V = 1/L\text{)}$$

Eq. 7.15 becomes: $$\frac{d\theta}{d\tau} + a \cdot \theta - b = 0 \qquad ...(7.16)$$

Now, introduce the transformation: $\theta' = \theta - \dfrac{b}{a}$ ...(a)

then, $$\frac{d\theta'}{d\tau} = \frac{d\theta}{d\tau}$$

and, substituting Eq. a in 7.16:

$$\frac{d\theta'}{d\tau} + a \cdot \theta' = 0 \qquad ...(7.17)$$

Seperating the variables and intergrating from $\tau = 0$ to $\tau = \tau$, (and, $\theta' = \theta_i'$ to $\theta' = \theta'$)

$$\frac{\theta'}{\theta_i'} = \exp(-a \cdot \tau) \qquad ...(7.18)$$

Substituting now for θ' and θ:

$$\frac{T(\tau) - T_a - \left(\frac{b}{a}\right)}{T_i - T_a - \left(\frac{b}{a}\right)} = \exp(-a \cdot \tau) \qquad ...(7.19)$$

i.e.

$$\frac{T(\tau) - T_a}{T_i - T_a} = \exp(-a \cdot \tau) + \frac{\frac{b}{a}}{T_i - T_a} \cdot (1 - \exp(-a \cdot \tau)) \qquad ...(7.20)$$

and, also from Eq. 7.19:

$$\tau = \frac{-1}{a} \cdot \ln\left[\frac{T(\tau) - T_a - \left(\frac{b}{a}\right)}{T_i - T_a - \left(\frac{b}{a}\right)}\right] \qquad ...(7.21)$$

Note that for $\tau = \infty$, Eq. 7.20 reduces to:

$$T(\tau) = T_a + \frac{b}{a} = T_a + \frac{q}{h} \qquad ...(7.22)$$

Eq. 7.22 gives the steady state temperature in the slab.

Example 7.5. An aluminium plate (ρ = 2707 kg/m^3, C_p = 0.896 kJ/(kgC), and k = 200 W/(mC)) of thickness 3 cm is at an initial, uniform temperature of 60°C. Suddenly, it is subjected to uniform heat flux q = 8000 W/m^2, on one surface while the other surface is exposed to an air stream at 25°C, with a heat transfer coefficient of h = 50 W/(m^{2}C).

(i) Is lumped system analysis applicable to this case?
(ii) If yes, plot the temperature of the plate as a function of time, and
(iii) What is the temperature of the plate in steady state?

Solution. See Fig. Example 7.5.

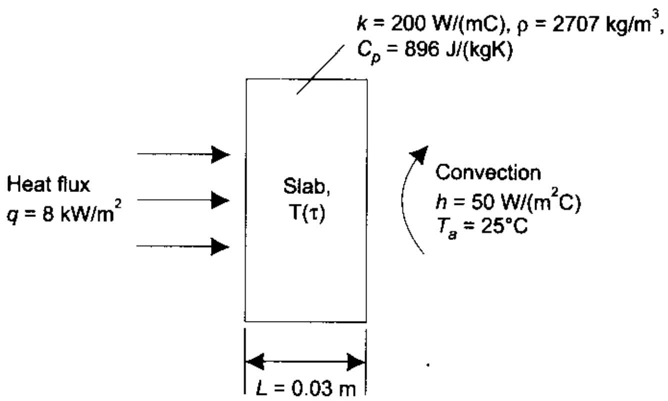

Figure Example 7.5 Transient conduction in a slab with mixed boundary conditions

Data:

$L := 0.03$ m $\quad \rho := 2707$ kg/m^3 $\quad C_p := 896$ J/(kgC) $\quad k := 200$ W/(mC) $\quad T_i := 60$°C $\quad T_a := 25$°C

$h := 50$ W/(m^{2}C) $\quad q := 8000$ W/m^2

First, claculate the Biot number:

$$Bi = \frac{h \cdot L_c}{k} = \frac{h}{k} \cdot \left(\frac{V}{2 \cdot A}\right) = \frac{h \cdot \frac{L}{2}}{k}$$ *(definition of Biot number)*

i.e. $Bi := \frac{h \cdot L}{2 \cdot k}$ i.e. $Bi = 3.75 \times 10^{-3}$ ...***Biot number***

Since $Bi < 0.1$, **lumped system analysis is applicable**, and the temperature variation within the solid will be within an error of 5%.

To plot the temperature of plate as a function of time:

Clearly, this is a case of mixed boundary conditions, wherein at the left surface there is heat input by uniform heat flux impinging on that surface, and on the right surface, there is heat loss by convection. So, we can directly apply Eq. 7.20.

i.e. $$\frac{T(\tau) - T_a}{T_i - T_a} = \exp(-a \cdot \tau) + \frac{\frac{b}{a}}{T_i - T_a} \cdot (1 - \exp(-a \cdot \tau)) \quad ..(7.20)$$

Here, $$a = \frac{h \cdot A}{\rho \cdot V \cdot C_p}$$

and, $$b = \frac{q \cdot A}{\rho \cdot V \cdot C_p}$$ (remember: $A/V = 1/L$)

i.e. $$a := \frac{h}{\rho \cdot C_p \cdot L}$$ i.e. $a = 6.872 \times 10^{-4}$

and, $$b := \frac{q}{\rho \cdot C_p \cdot L}$$ i.e. $b = 0.11$

Therefore, from Eq. 7.20:

$$T(\tau) := T_a + (T_i - T_a) \cdot \left[\exp(-a \cdot \tau) + \frac{\frac{b}{a}}{T_i - T_a} \cdot (1 - \exp(-a \cdot \tau)) \right] \quad \text{...define } T(\tau)$$

To plot $T(\tau)$ against time, let us define a range variable τ, from say 0 s to 10,000 s, at an interval of 50 s. Then, select the x–y plot from the graph palette, and fill up the place holders on the x-axis and y-axis with τ and $T(\tau)$, respectively. Click anywhere outside the graph and immediately the graph appears: See Fig. Ex. 7.5(b)

$\tau := 0, 50, \ldots, 10{,}000$ *(define a range variable τ, such that initial value = 0, next value = 50 and last value = 1000 s.)*

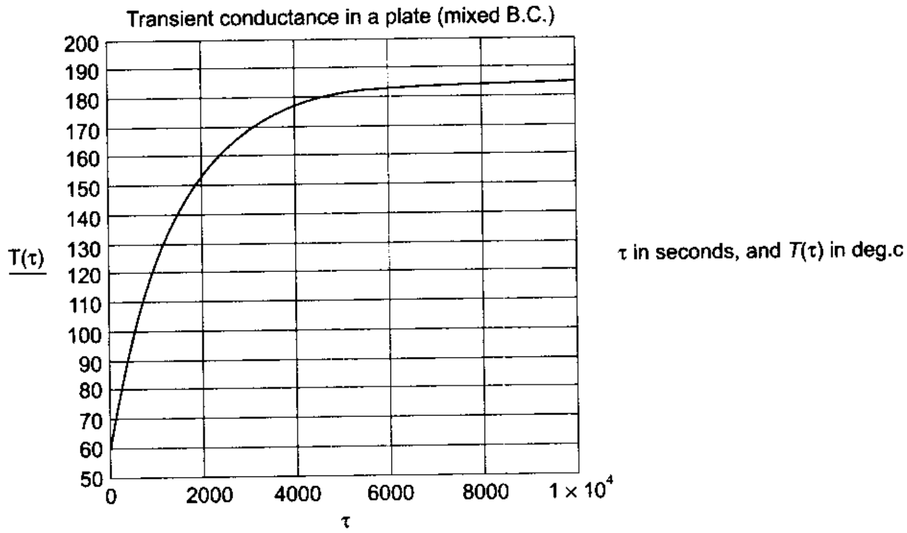

Figure Example 7.5(b)

Temperature of plate in steady state:

We directly use Eq. 7.22 for $\tau = \infty$, i.e. steady state condition:

i.e. $$T_{steady} = T_a + \frac{q}{h} \quad ...(7.22)$$

i.e. $T_{steady} = 185°C$ ***(steady state temperature of plate.)***

Note from the above graph, that at large τ (= beyond about 8,000 s), the temperature of the plate, indeed, tends to a value of 185°C.

Example 7.6. A household electric iron has an aluminium base (ρ = 2700 kg/m^3, C_p = 0.896 kJ/(kgC), and k = 200 W/(mC)), and weighs 1.5 kg. Total area of iron is 0.06 m^2 and it is heated with a 500 W heating element. Initially, the iron is at ambient temp. of 25°C. How long will it take for the iron to reach 110°C once it is switched on? Take heat transfer coefficient between iron and the ambient air as 15 W/(m^{2}K).

Solution.

Data:

$A := 0.06$ m^2 $\quad \rho := 2700$ kg/m^3 $\quad C_p := 896$ J/(kgC) $\quad k := 200$ W/(mC) $\quad T_i := 25°C \quad T_a := 25°C$

$h := 15$ W/(m^{2}C) $\quad T := 110°C \quad M := 1.5$ kg $\quad Q := 500$ W

First, calculate the Biot number:

$V := \frac{M}{\rho}$ i.e. $V = 5.556 \times 10^{-4}$, m^3 *(volume of iron)*

$$Bi = \frac{h \cdot L_c}{k} = \frac{h}{k} \cdot \left(\frac{V}{A}\right)$$ *(definition of Biot number)*

$$Bi := \frac{h}{k} \cdot \frac{\left(\frac{M}{\rho}\right)}{A}$$ *(since Volume of iron = Mass/density)*

i.e. $Bi = 6.944 \times 10^{-4}$ ***(Biot number.)***

Since $Bi < 0.1$, **lumped system analysis is applicable**, and the temperature variation within the solid will be within an error of 5%.

Now, writing the energy balance for the iron at any time τ,

Rate of total heat generated – Rate of heat lost by convection = Rate of increase of internal energy

i.e. $$Q - h \cdot A \cdot (T(\tau) - T_a) = \rho \cdot V \cdot C_p \cdot \frac{dT(\tau)}{d\tau} \quad ...(a)$$

i.e. $$\frac{dT(\tau)}{d\tau} + \frac{h \cdot A \cdot (T(\tau) - T_a)}{\rho \cdot V \cdot C_p} - \frac{Q}{\rho \cdot V \cdot C_p} = 0 \quad ...(b)$$

Substituting: $\theta = T(\tau) - T_a$ i.e. $\frac{d\theta}{d\tau} = \frac{dT(\tau)}{d\tau}$

and, let: $$a = \frac{h \cdot A}{\rho \cdot V \cdot C_p}$$

and, $$b = \frac{Q}{\rho \cdot V \cdot C_p}$$

Eq. b becomes: $$\frac{d\theta}{d\tau} + a \cdot \theta - b = 0 \quad ...(c)$$

Note that Eq. c is the same as Eq. 7.16, derived earlier. And, the solution for τ is obtained as Eq. 7.21, with the definition of 'a' and 'b' as follows:

$$a := \frac{h \cdot A}{\rho \cdot V \cdot C_p} \quad \text{i.e.} \quad a = 6.696 \times 10^{-4}$$

and, $$b := \frac{Q}{\rho \cdot V \cdot C_p} \quad \text{i.e.} \quad b = 0.372$$

And, $\tau := \frac{-1}{a} \cdot \ln \left[\frac{T - T_a - \frac{b}{a}}{T_i - T_a - \frac{b}{a}} \right]$ ...(7.21)

i.e. $\tau = 247.975$ s ***...time required for iron to reach* 110°C.**

7.6 One-dimensional Transient Conduction in Large Plane Walls, Long Cylinders and Spheres when Biot Number > 0.1

There are many situations in practice, where the temperature gradient in the solid is not negligible (i.e. $Bi > 0.1$) and the lumped system analysis is not applicable. In such situations, we start with the general differential equation for time dependent, one-dimensional conduction in the appropriate coordinate system and solve it in conjunction with the boundary conditions.

In this section, we shall analyse one-dimensional transient conduction in large plane walls, long cylinders and spheres when $Bi > 0.1$.

7.6.1 One Term Approximation Solutions

Fig. 7.6 shows schematic diagram and coordinate systems for a large, plane slab, long cylinder and a sphere.

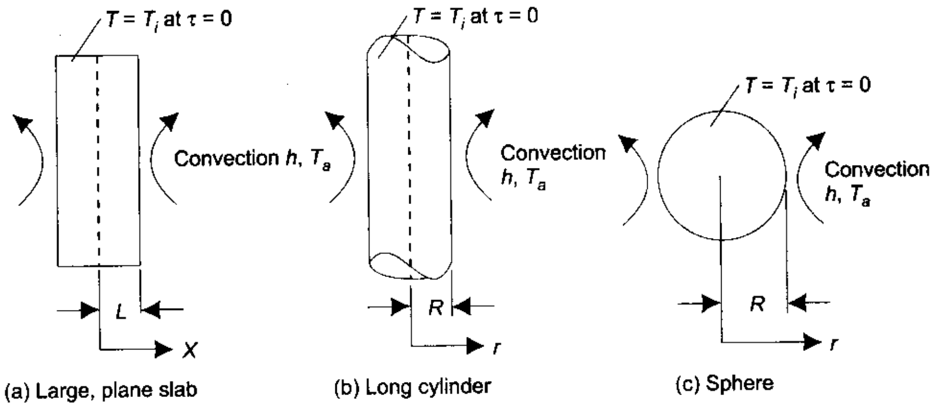

FIGURE 7.6 One-dimensional transient conduction in simple geometries

Consider a plane slab of thickness $2L$, shown in Fig. 7.6(a) above. Initially, i.e. at $\tau = 0$, the slab is at an uniform temperature, T_i. Suddenly, at $\tau = 0$, both the surfaces of the slab are subjected to convection heat transfer with an ambient at temperature T_a, with a heat transfer coefficient of h, as shown. Since there is geometrical and thermal symmetry, we need to consider only half the slab, and that is the reason why we chose the origin of the coordinate system on the mid-plane. Then, we can write the mathematical formulation of the problem for plane slab as follows:

$$\frac{d^2T}{dx^2} = \frac{1}{\alpha} \cdot \frac{dT}{d\tau} \quad \text{in } 0 < x < L, \text{ for } \tau > 0 \qquad ...(7.23, a)$$

$$\frac{dT}{dx} = 0 \quad \text{at } x = 0, \text{ for } \tau > 0 \qquad ...(7.23, b)$$

$$-k \cdot \frac{dT}{dx} = h \cdot (T - T_a) \quad \text{at } x = L, \text{ for } \tau > 0 \qquad ...(7.23, c)$$

$$T = T_i \quad \text{for } \tau = 0, \text{ in } 0 < x < L \qquad ...(7.23, d)$$

The solution of the above problem, however, is rather involved and consists of infinite series. So, it is more convenient to present the solution either in tabular form or charts.

For this purpose, we define the following dimensionless parameters:

(i) Dimensionless temperature:

$$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a}$$

(ii) Dimensionless distance from the centre:

$$X = \frac{x}{L}$$

(iii) Dimensionless heat transfer coefficient:

$$Bi = \frac{h \cdot L}{k} \qquad \textbf{\textit{(Biot number)}}$$

(iv) Dimensionless time:

$$F_o = \frac{\alpha \cdot \tau}{L^2} \qquad \textbf{\textit{(Fourier number)}}$$

Non-dimensionalisation of the results with the above-mentioned dimensionless numbers enables us to present the results practically over a wide range of operating parameters, either in tabular or graphical forms.

To deal with a long cylinder or a sphere, we do exactly what we did with the plane slab, i.e. start with the appropriate differential equation for one-dimensional, time dependent conduction in cylindrical or spherical coordinates. Boundary conditions will be the same as in Eq. 4.23 except that x is replaced by r and L is replaced by R. Again, results are non-dimensionalised with the dimensionless parameters as mentioned above; however, **note one important difference in defining Biot number now, while using the tabular or chart solutions**:

Characteristic length in Biot number is taken as half-thickness L for a plane wall, Radius R for a long cylinder and sphere instead of being calculated as V/A, as done in lumped system analysis.

For all these three geometries, as mentioned earlier, the solution involves infinite series, which are difficult to deal with. However, it is observed that for $Fo > 0.2$, considering only the first term of the series and neglecting other terms, involves an error of less than 2%. Generally, we are interested in times, $Fo > 0.2$. So, it becomes very useful and convenient to use **one term approximation solution**, for all these three cases, as follows:

Plane wall:
$$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\frac{\lambda_1 \cdot x}{L}\right) \qquad ...Fo > 0.2 \qquad ...(7.24, a)$$

Long cylinder:
$$\theta(x, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot J_0\left(\frac{\lambda_1 \cdot r}{R}\right) \qquad ...Fo > 0.2 \qquad ...(7.24, b)$$

Sphere:
$$\theta(x, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \frac{\sin\left(\frac{\lambda_1 \cdot r}{R}\right)}{\frac{\lambda_1 \cdot r}{R}} \qquad ...Fo > 0.2 \qquad ...(7.24, c)$$

In the above equations, A_1 and λ_1 are functions of Biot number only.

A_1 and λ_1 are calculated from the following relations:

For Plane wall:

$$\lambda_1 \cdot \tan(\lambda_1) = Bi$$

$$A_1 = \frac{4 \cdot \sin(\lambda_1)}{2 \cdot (\lambda_1) + \sin[2 \cdot (\lambda_1)]}$$

For Long cylinder:

$$\lambda_1 \cdot \frac{J_1(\lambda_1)}{J_0(\lambda_1)} = Bi$$

$$A_1 = \frac{2 \cdot J_1(\lambda_1)}{\lambda_1 \cdot [(J_0(\lambda_1))^2 + (J_1(\lambda_1))^2]}$$

For Sphere:

$$1 - \lambda_1 \cdot \cot(\lambda_1) = Bi$$

$$A_1 = \frac{4 \cdot [\sin(\lambda_1) - (\lambda_1) \cdot \cos(\lambda_1)]}{2 \cdot (\lambda_1) - \sin[2 \cdot (\lambda_1)]}$$

Values of A_1 and λ_1 are given in Table 7.1. (See Appendix at the end of this chapter for Mathcad functions to calculate these parameters). Function J_0 is the zeroth order Bessel function of the first kind and J_1 is the first order Bessel function of the first kind. Values of J_0 and J_1 can be read from Table 7.2. (Obtained directly from Mathcad).

Now, at the centre of a plane wall, cylinder and sphere, we have the condition $x = 0$ or $r = 0$. Then, noting that $\cos(0) = 1$, $J_0(0) = 1$, and limit of $\sin(x)/x$ is also 1, Eq. 7.24 becomes:

at the centre of plane wall, cylinder and sphere:

Centre of plane wall: $(x = 0)$

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \qquad ...(7.25, a)$$

Centre of long cylinder: $(r = 0)$

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \qquad ...(7.25, b)$$

Centre of sphere: $(r = 0)$

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \qquad ...(7.25, c)$$

Therefore, first step in the solution is to calculate the Biot number; once the Biot number is known, constants A_1 and λ_1 are found out from Tables 7.1 and 7.2, and then use relations given in Eqs. 7.24 and 7.25 to calculate the temperature at any desired location.

The one-term solutions are presented in chart form in the next section. But, generally, it is difficult to read charts accurately. So, relations given in Eqs. 7.24 and 7.25 along with Tables 7.1 and 7.2, should be preferred to the charts.

Calculation of amount of heat transferred, Q:

Many times, we need to calculate the amount of heat lost (or gained) by the body, Q, during the time interval $\tau = 0$ to $\tau = \tau$, i.e. from the beginning up to a given time. Again, we non-dimensionalise Q by dividing it by Q_{max}, the maximum possible heat transfer. Obviously, maximum amount of heat has been transferred when the body has reached equilibrium with the ambient, i.e.

$$Q_{max} = \rho \cdot V \cdot C_p \cdot (T_i - T_a) = m \cdot C_p \cdot (T_i - T_a) \qquad ...(7.26)$$

where ρ is the density, V is the volume, (ρV) is the mass, C_p is the specific heat of the body.

If Q_{max} is positive, body is losing energy; and if it is negative, body is gaining energy.

Based on the one-term approximation discussed above, (Q/Q_{max}) is calculated for the three cases, from the following:

Plane wall:

$$\frac{Q}{Q_{max}} = 1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} \qquad ...(7.27, a)$$

Cylinder:

$$\frac{Q}{Q_{max}} = 1 - 2 \cdot \theta_0 \cdot \frac{J_1(\lambda_1)}{\lambda_1} \qquad ...(7.27, b)$$

Sphere:

$$\frac{Q}{Q_{max}} = 1 - 3 \cdot \theta_0 \cdot \left(\frac{\sin(\lambda_1) - \lambda_1 \cdot \cos(\lambda_1)}{\lambda_1^3}\right) \qquad ...(7.27, c)$$

Note:

(i) Remember well the definition of Biot number i.e. $Bi = (hL/k)$, where L is half-thickness of the slab, and $Bi = (hR/k)$, where R is the outer radius of the cylinder or the sphere.

(ii) Foregoing results are equally applicable to a plane wall of thickness L, insulated on one side and suddenly subjected to convection at the other side. This is so because, the boundary condition $dT/dx = 0$

TABLE 7.1 Transient heat conduction in a plane wall, long cylinder and sphere-coefficients for one-term approximation

	Plane wall		Cylinder		Sphere	
Bi,	λ_1,	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

at $x = 0$ for the mid-plane of a slab of thickness $2L$ (see Eq. 7.23, b), is equally applicable to a slab of thickness L, insulated at $x = 0$.

(iii) These results are also applicable to determine the temperature response of a body when temperature of its surface is suddenly changed to T_s. This case is equivalent to having convection at the surface with a heat transfer coefficient,

$h = \infty$; now, T_a is replaced by the prescribed surface temperature, T_s.

(iv) Again, remember that these results are valid for the situation where Fourier number, $Fo > 0.2$.

7.6.2 Heisler and Grober Charts

The one term approximation solutions (Eq. 7.25) were represented in graphical form by Heisler in 1947. They were supplemented by Grober in 1961, with graphs for heat transfer Eq. 7.27. These graphs are shown in Fig. 7.7, 7.8 and 7.9, for plane wall, long cylinder and a sphere, respectively.

TABLE 7.2 Zeroth and first order Bessel functions of the first kind

$x := 0, 0.1, \ldots, 3.2$ ...*define range variable x from 0 to 3.2, with an increment of 0.1*

x	$J_0(x)$	$J_1(x)$
0	1	0
0.1	0.9975	0.04994
0.2	0.99002	0.0995
0.3	0.97763	0.14832
0.4	0.9604	0.19603
0.5	0.93847	0.24227
0.6	0.912	0.2867
0.7	0.8812	0.329
0.8	0.84629	0.36884
0.9	0.80752	0.40595
1	0.7652	0.44005
1.1	0.71962	0.4709
1.2	0.67113	0.49829
1.3	0.62009	0.52202
1.4	0.56686	0.54195
1.5	0.51183	0.55794
1.6	0.4554	0.5699
1.7	0.39798	0.57777
1.8	0.33999	0.58152
1.9	0.28182	0.58116
2	0.22389	0.57762
2.1	0.16661	0.56829
2.2	0.11036	0.55596
2.3	0.05554	0.53987
2.4	0.00251	0.52019
2.5	−0.04838	0.49708
2.6	−0.0968	0.47082
2.7	−0.14245	0.4416
2.8	−0.18504	0.40971
2.9	−0.22431	0.37543
3	−0.26005	0.33906
3.1	−0.29206	0.30092
3.2	−0.32019	0.26134

How to use these charts?

First chart in each of these figures gives the non-dimensionalised centre temperature T_0. i.e. at $x = 0$ for the slab of thickness $2L$, and at $r = 0$ for the cylinder and sphere, at a given time τ. Temperature at any other position at the same time τ, is calculated using the next graph, called 'position correction chart'. Third chart gives Q/Q_{max}.

Procedure of using these charts to solve a numerical problem is as follows:

(i) First of all, calculate Bi from the data, with the usual definition of Bi, i.e. $Bi = (h.L_c)/k$, where L_c is the characteristic dimension, given as: $L_c = (V/A)$ i.e. $L_c = L$, half-thickness for a plane wall, $L_c = R/2$ for a cylinder, and $L_c = R/3$ for a sphere. If $B_i < 0.1$, use lumped system analysis; otherwise, go for one-term approximation or chart solution.

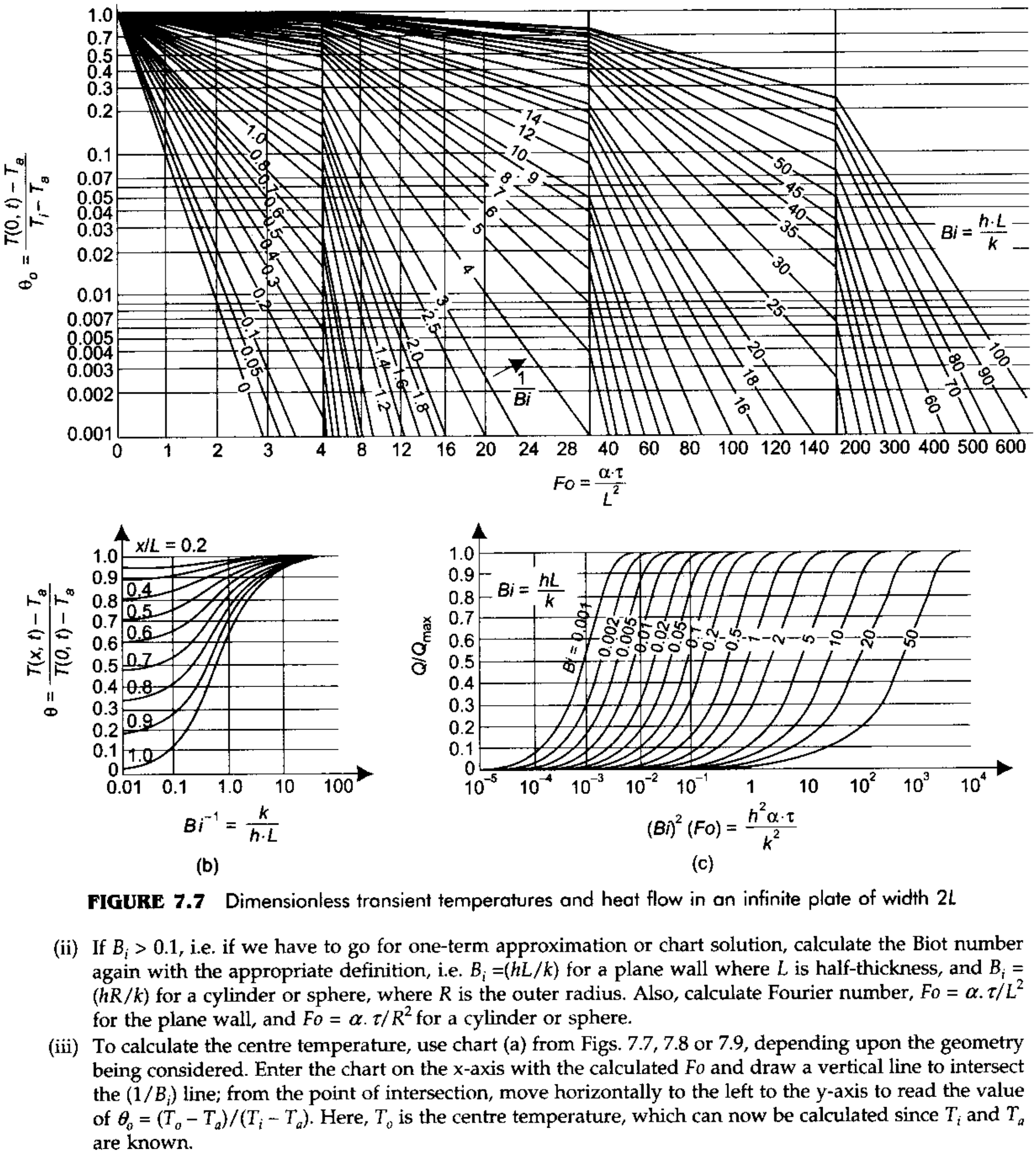

FIGURE 7.7 Dimensionless transient temperatures and heat flow in an infinite plate of width 2L

(ii) If $B_i > 0.1$, i.e. if we have to go for one-term approximation or chart solution, calculate the Biot number again with the appropriate definition, i.e. $B_i = (hL/k)$ for a plane wall where L is half-thickness, and $B_i = (hR/k)$ for a cylinder or sphere, where R is the outer radius. Also, calculate Fourier number, $Fo = \alpha.\tau/L^2$ for the plane wall, and $Fo = \alpha.\tau/R^2$ for a cylinder or sphere.

(iii) To calculate the centre temperature, use chart (a) from Figs. 7.7, 7.8 or 7.9, depending upon the geometry being considered. Enter the chart on the x-axis with the calculated Fo and draw a vertical line to intersect the $(1/B_i)$ line; from the point of intersection, move horizontally to the left to the y-axis to read the value of $\theta_o = (T_o - T_a)/(T_i - T_a)$. Here, T_o is the centre temperature, which can now be calculated since T_i and T_a are known.

(iv) To calculate the temperature at any other position, use Fig. b of Fig. 7.7, 7.8 or 7.9, as appropriate. Enter the chart with $1/B_i$ on the x-axis, move vertically up to intersect the (x/L) or (r/R) curve as the case may be, and from the point of intersection, move to the left to read on the y-axis, the value of $\theta = (T - T_a)/(T_o - T_a)$. Here, T is the desired temperature at the indicated position. We multiply θ and θ_o to get:

i.e.
$$\theta \cdot \theta_0 = \left(\frac{T - T_a}{T_0 - T_a}\right) \cdot \left(\frac{T_0 - T_a}{T_i - T_a}\right) = \frac{T - T_a}{T_i - T_a} \quad \text{...(7.28)}$$

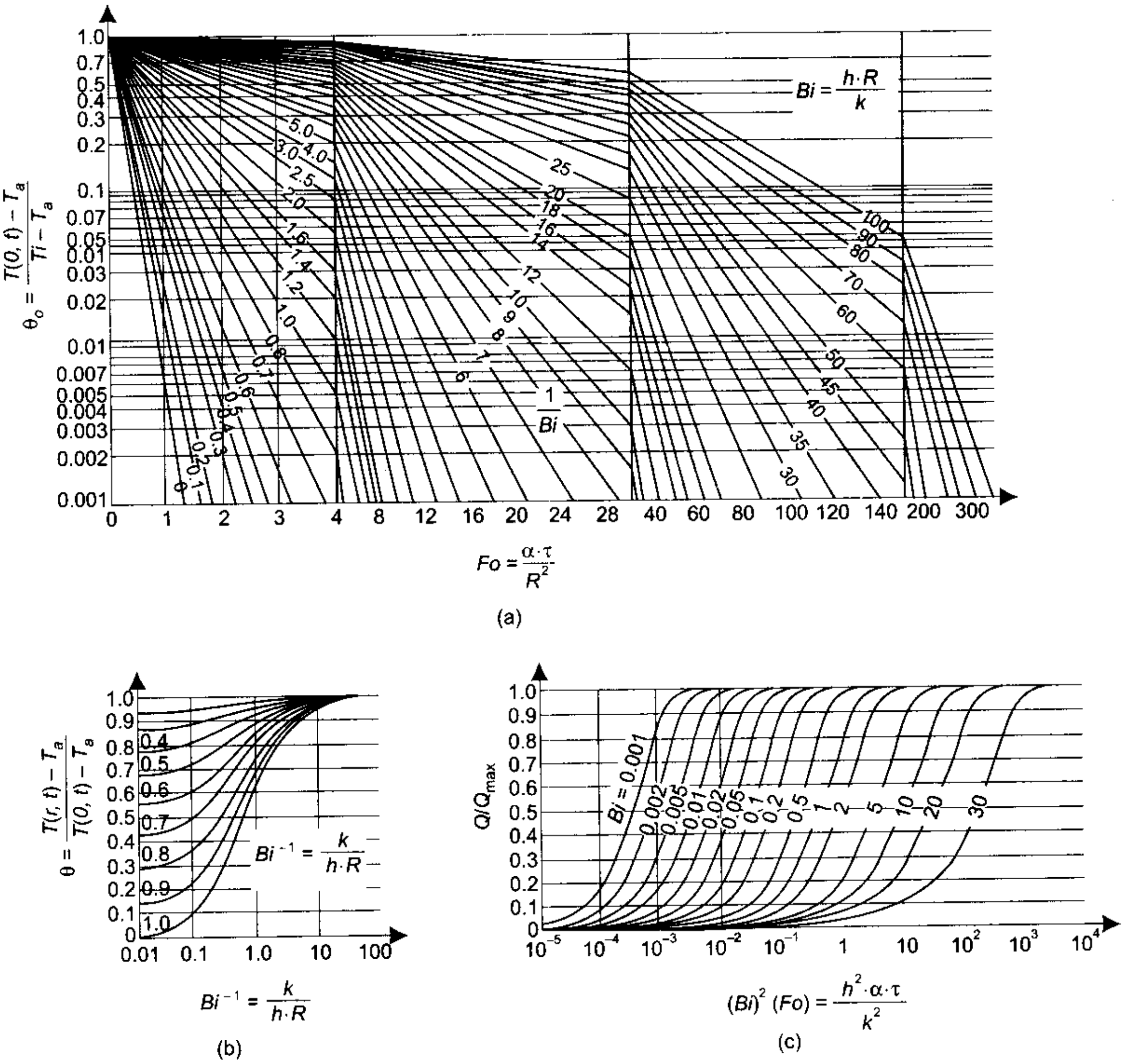

FIGURE 7.8 Dimensionless transient temperatures and heat flow for a long cylinder

From Eq. 7.28, we can easily calculate T, the desired temperature at the given position, since T_i and T_a are known.

(v) To find out the amount of heat transferred Q, during a particular time interval τ from the beginning (i.e. $\tau = 0$), use Fig. c from Figs. 7.7, 7.8 or 7.9, depending upon the geometry. Enter the x-axis with the value of $(B_i^2 . Fo)$ and move vertically up to intersect the curve representing the appropriate B_i, and move to the left to read on the y-axis, the value of Q/Q_{max}. Q is then easily found out since $Q_{max} = mC_p(T_i - T_a)$. And, $Q = (Q/Q_{max}) . Q_{max}$.

Note the following in connection with these charts:

(i) These charts are valid for Fourier number $Fo > 0.2$.

(ii) Specifically, remember that while calculating Biot number, characteristic length (L_c) used is L, the half-thickness for a plane wall, and outer radius, R for the cylinder and the sphere (L_c is, now, not equal to: (V/A)).

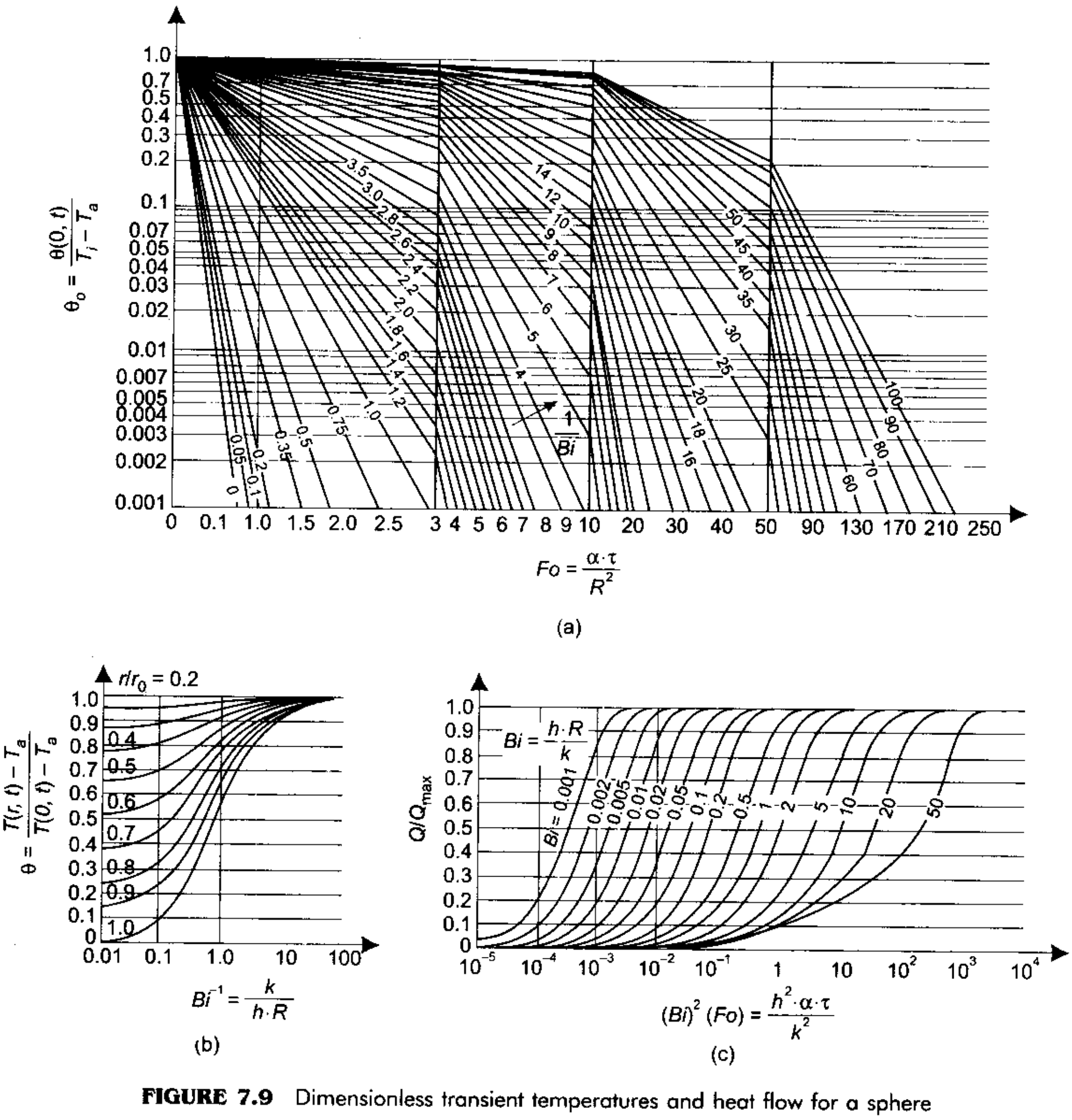

FIGURE 7.9 Dimensionless transient temperatures and heat flow for a sphere

(iii) In these graphs, $(1/B_i) = 0$, corresponds to $h \to \infty$, which means that at $\tau = 0$, the surface of the body is suddenly brought to a temperature of T_a and thereafter maintained at T_a at all times.

(iv) To calculate Q up to a given time, first find out Q/Q_{max} from the Grober's chart and calculate Q_{max} from $Q_{max} = mC_p(T_i - T_a)$. (See Eq. 7.26). Then, Q is calculated as: $Q = (Q/Q_{max}).Q_{max}$.

(v) Note from the 'position correction charts' that at $B_i < 0.1$ (i.e. $1/B_i > 10$), temperature within the body can be taken as uniform, without introducing an error of more than 5%. This was precisely the condition for application of 'lumped system analysis'.

(vi) As stated earlier, it is difficult to read these charts accurately, since logarithmic scales are involved; also, the graphs are rather crowded with lines. So, use of one-term approximation with tabulated values of A_1 and λ_1 should be preferred. However, these graphs are extremely useful for a quick estimation of values required.

Use of one-term approximation solutions and the transient conduction charts is illustrated in the following examples.

Example 7.7. A steel plate ($\alpha = 1.2 \times 10^{-5}$ m²/s, k = 43 W/(mC)), of thickness $2L$ = 10 cm, initially at a uniform temperature of 250°C is suddenly immersed in an oil bath at T_a = 45°C. Convection heat transfer coefficient between the fluid and the surfaces is 700 W/(m²C).

(i) How long will it take for the centre plane to cool to 100°C?

(ii) What fraction of the energy is removed during this time?

(iii) Draw the temperature profile in the slab at different times.

Solution.

Data:

$L := 0.05$ m $\quad \alpha := 1.2 \cdot 10^{-5}$ m²/s $\quad k := 43$ W/(mC) $\quad T_i := 250°C \quad T_a := 45°C$

$h := 700$ W/(m²C) $\quad T_0 := 100°C$

To calculate: the time τ, surface temperature and fraction of heat transferred Q/Q_{max}.

First, check if lumped system analysis is applicable:

$$Bi := \frac{h \cdot L}{k} \qquad \textit{(define Biot number)}$$

i.e. $\quad Bi = 0.814 \qquad$ *(Biot number.)*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable**.

We will adopt Heisler chart solution and then check the results from one-term approximation solution.

To find the time required for the centre to reach 100°C:

For using the charts, $B_i = hL/k$, which is already calculated.

Fourier number: $\quad Fo = \dfrac{\alpha \cdot \tau}{L^2}$

Centre temperature is given as 100°C. Therefore, calculate θ_0:

$$\theta_0 := \frac{T_0 - T_a}{T_i - T_a} \qquad \textit{(define } \theta_0\textit{)}$$

i.e. $\quad \theta_0 = 0.268 \qquad$ (value of θ_0)

Also, $\quad \dfrac{1}{Bi} = 1.229 \qquad$ *(value of 1/Bi)*

Now, with this value of θ_0, enter the y-axis of Fig. 7.7,a. Move horizontally to intersect the $1/B_i$ = 1.229 line; from the point of intersection, move vertically down to x-axis to read Fo = 2.4.

So, we get: $\quad Fo := 2.4$

Then, $\quad \tau := \dfrac{Fo \cdot L^2}{\alpha}$ s $\qquad$ *(define τ, the time required for the centre to reach* 100°C)

i.e. $\quad \tau$ = 500s $\qquad$ ***(time required for the centre to reach* 100°C.)**

Surface temperature:

At the surface, x/L =1. Enter Fig. 7.7, b on the x-axis with a value of $1/B_i$ = 1.229, move up to intersect the curve of x/L = 1, then move to left to read on y-axis the value of θ = 0.7

i.e. $\quad \theta = \dfrac{T - T_a}{T_0 - T_a} = 0.7$

Therefore, $\quad T := 0.7 \cdot (T_0 - T_a) + T_a$ °C $\qquad$ *(temperature on the surface)*

i.e. $\quad T$ = 83.5°C $\qquad$ ***(temperature on the surface.)***

Fraction of maximum heat transferred, Q/Q_{max}:

We will use Grober's chart, Fig. 7.7, c:

We need $B_i^2 Fo$ to enter the x-axis:

We get: $\quad B_i^2 \cdot Fo = 1.59$

With the value of 1.59, enter the x-axis of Fig. 7.7, c, move vertically up to intersect the curve of B_i = 0.814, then move horizontally to read Q/Q_{max} = 0.8

i.e. from Fig. 7.7c, we get: $\dfrac{Q}{Q_{max}} = 0.8$

i.e. 80% of the energy is removed by the time the centre tamperature reached 100°C.

Verify by one-term approximation solution:

Time required for the centre to reach 100°C:

From Eq. 7.25, a, we have:

$$\text{Centre of plane wall: } \theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \qquad \text{...(7.25, a)}$$

$(x = 0)$

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 0.814$

Interpolating: $\lambda_1 = 0.7910 + \dfrac{0.8274 - 0.7910}{10} \cdot 1.4$

i.e. $\lambda_1 = 0.796$

and, $A_1 = 1.1016 + \dfrac{1.1107 - 1.1016}{10} \cdot 1.4$

i.e. $A_1 = 1.103$

Now, $\dfrac{T_0 - T_a}{T_i - T_a} = 0.26829$

Therefore, Eq. 7.25, a becomes:

$$\frac{0.26829}{1.103} = e^{-0.796^2 \cdot Fo}$$

i.e. $$Fo := \frac{-\ln\left(\frac{0.268}{1.103}\right)}{(0.796)^2}$$

i.e. $Fo = 2.233$

Then, $\tau := \dfrac{Fo \cdot L^2}{\alpha}$ s *(define τ, the time required for the centre to reach 100°C)*

i.e. $\tau = 465.188$ s ***(time required for the centre to reach*** **100°C.)**

Compare this value with the one got from Heisler's chart, i.e. 500 s. The error is in reading the chart.

Surface temperature when the centre has reached 100°C:

From Eq. 7.24, a, we have:

$$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\frac{\lambda_1 \cdot x}{L}\right) \quad \text{...}Fo > 0.2 \qquad \text{...(7.24, a)}$$

Here, $x/L = 1$, at the surface of the plate. So, we get:

$T := (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos(\lambda_1)\right) + T_a$ *(define $T(x, \tau)$)*

$T = 83.413$°C ***(temperature at the surface.)***

Compare this with the value of 83.5°C obtained earlier. They are quite close.

Fraction of maximum heat transferred, Q/Q_{max}:

From Eq. 7.27, a, we have:

$$\text{Plane wall: } \frac{Q}{Q_{max}} = 1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} \qquad \text{...(7.27, a)}$$

i.e. $\text{Fraction} := 1 - \dfrac{T_0 - T_a}{T_i - T_a} \cdot \dfrac{\sin(\lambda_1)}{\lambda_1}$ *...define Fraction, Q/Q_{max}*

i.e. Fraction = 0.759

i.e. 75.9% of the energy is removed by the time the centre temperature has reached 100°C.

Compare this with the value of 80% obtained earlier; again, the error is in reading the charts.

Note: It is apparent from this example that the error involved in reading the graphs can be substantial; this is because logarithmic scales are involved and also the lines are rather crowded in the graph. So, one-term approximation with table of values of A_1 and λ_1 against B_i should be preferred.

To draw temperature profile in the plate at different times:

We have, for temperature distribution at any location:

Plane wall: $\theta(x, \tau) = \dfrac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\dfrac{\lambda_1 \cdot x}{L}\right)$ ...$Fo > 0.2$ ...(7.24, a)

And, centre of plane wall: $\theta_0 = \dfrac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ ...(7.25, a)
($x = 0$)

Fourier number as a function of τ: $Fo(\tau) := \dfrac{\alpha \cdot \tau}{L^2}$ ...*for slab*

By writing Fourier number as a function of τ, and including it in Eq. A below as shown, it is ensured that for each new τ, the corresponding new Fo is calculated.

Then, $$T(x, \tau) := \begin{cases} T_a + (T_i - T_a)\cdot\left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo(\tau)}\right) & \text{if } x = 0 \\ T_a + (T_i - T_a)\cdot\left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo(\tau)} \cdot \cos\left(\dfrac{\lambda_1 \cdot x}{L}\right)\right) & \text{otherwise} \end{cases}$$ ...(A)

For a given τ, we will plot Eq. A against x; then, we will repeat for different times, τ.

We use Mathcad to draw the graph. First, define a range variable x, varying from 0 to say, 0.05 m, with an increment of 0.001 m. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with x and $T(x,30)$, respectively. Since our aim is to plot $T(x, \tau)$ for different values of x for given τ, start with $\tau = 30$ s; immediately, this graph is drawn, when we click anywhere outside the graph region. To get the graph for next value of $\tau = 120$, on the y-axis, next to the earlier entry, type a comma and enter $T(x,120)$ and click anywhere outside the graph region. Repeat this for different values of τ, as shown. See Fig. Example 7.7.

$x := 0, 0.001, \ldots, 0.05$ *(define a range variable x varying from zero to 0.05 m, with an increment of 0.001 m)*

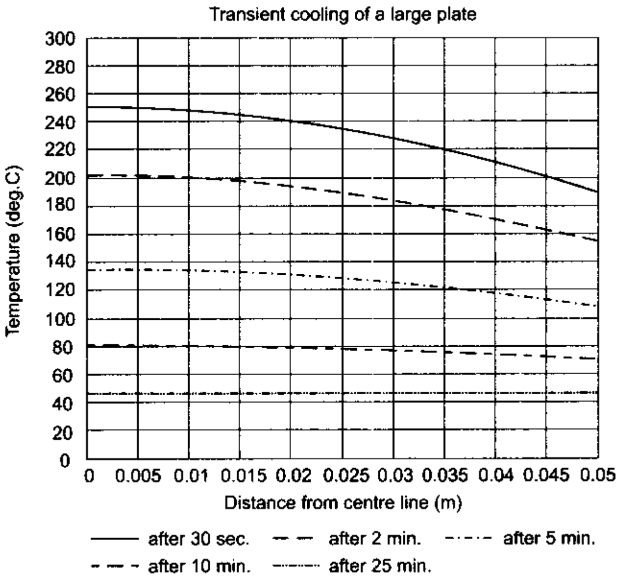

FIGURE Example 7.7 Transient cooling of a large plate, one-term approximation solution

Note:

(i) Note that the above graph shows temperature distribution for one half of the plate; for the other half, the temperature distribution will be identical.

(ii) See from the above Fig. Example 7.7 how cooling progresses with time. After a time period of 25 min the temperatures in the plate are almost uniform at 45°C.

(iii) Eq. A illustrates a small piece of Mathcad programming. It uses the "if...otherwise" condition, i.e. if $x = 0$, the temperature at the centre is given by Eq.7.25, a; otherwise, temperature distribution is given by Eq. 7.24, a.

Example 7.8. A long, 15 cm diameter cylindrical shaft made of stainless steel 304 (k = 14.9 W/(mC), ρ = 7900 kg/m^3, C_p = 477 J/(kgC), and $\alpha = 3.95 \times 10^{-6}$ m^2/s), comes out of an oven at an uniform temperature of 450°C. The shaft is then allowed to cool slowly in a chamber at 150°C with an average heat transfer coefficient of 85 W/(m^{2}C).

(i) Determine the temperature at the centre of the shaft 25 min after the start of the cooling process.
(ii) Determine the surface temperature at that time, and
(iii) Determine the heat transfer per unit length of the shaft during this time period.
(iv) Draw the temperature profile along the radius for different times.

Solution.

Data:

$L := 1$ m $\quad R := 0.075$ m $\quad \alpha := 3.95 \times 10^{-6}$ m^2/s $\quad k := 14.9$ W/(mC) $\quad C_p := 477$ J/(kg C) $\quad \rho := 7900$ kg/m^3

$T_i := 450$°C $\quad T_a := 150$°C $\quad h := 85$ W/(m2C) $\quad \tau := 1500$ s

To calculate: the centre temperature afer time τ, surface temperature and amount of heat transferred during this period.

First check if lumped system analysis is applicable:

$$Bi := \frac{h \cdot \frac{R}{2}}{k}$$ *(define Biot number...for a cylinder, $L_c = (V/A) = R/2$)*

i.e. $Bi = 0.214$ *(Biot number.)*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable.**

We will adopt Heisler chart solution and then check the results from one-term approximation solution.

To find the centre temperature after a time period of 1500 s:

For using the charts, now, remember that B_i is defined as:

$$Bi := \frac{h \cdot R}{k}$$ *(define Biot number)*

i.e. $Bi = 0.428$ *(Biot number)*

Fourier number: $$Fo := \frac{\alpha \cdot \tau}{R^2}$$ *(define Fourier number)*

i.e. $Fo = 1.053$ *(Fourier number)*

Also, $$\frac{1}{Bi} = 2.337$$ *...value of $1/Bi$*

Now, with the value of $Fo = 1.053$, enter the x-axis of Fig. 7.8,a. Move vertically up to intersect the $1/Bi = 2.337$ line; from the point of intersection, move horizontally to left, to read on the y-axis $\theta_0 = 0.49$.

So, we get: $$\theta_0 = 0.49 = \frac{T_0 - T_a}{T_i - T_a}$$

i.e. $T_0 := T_a + 0.49 \cdot (T_i - T_a)$ *(define centre temperature)*

i.e. $T_0 = 297$°C ***(centre temperature after 25 min duration.)***

Surface temperature:

At the surface, $r/R = 1$. Enter Fig. 7.8, b on the x-axis with a value of $1/Bi = 2.337$, move up to intersect the curve of $r/R = 1$, then move to left to read on y-axis th value of $\theta = 0.76$

i.e. $$\theta = \frac{T - T_a}{T_0 - T_a} = 0.76$$

Therefore, $T := 0.76 \cdot (T_0 - T_a) + T_a$°C ***(Temperature on the surface)***

i.e. $T = 261.72$°C ***(temperature on the surface.)***

Amount of heat transferred, Q:

We will use Grober's chart, Fig. 7.8, c:

We need $B_i^2 Fo$ to enter the x-axis:

We get: $B_i^2 \cdot Fo = 0.193$

With the value of 0.193, enter the x-axis of Fig. 7.8, c, move vertically up to intersect the curve of $B_i = 0.428$, then move horizontally to read $Q/Q_{max} = 0.55$

i.e. from Fig. 7.8, c, we get: $\frac{Q}{Q_{max}} = 0.55$

Now, $Q_{max} = \rho \cdot V \cdot C_p \cdot (T_i - T_a)$ *(maximum heat transfer possible)*

i.e. $Q_{max} := \rho \cdot (\pi \cdot R^2 \cdot L) \cdot C_p \cdot (T_i - T_a)$ *(define Q_{max})*

i.e. $Q_{max} = 1.998 \times 10^7$ J *(maximum heat transfer)*

Therefore, $Q = Q_{max} \cdot 0.55$ J *(define Q)*

i.e. $Q = 1.099 \times 10^7$ J ***(heat transferred during 25 min)***

Verify by one-term approximation solution:

Centre temperature reached after 25 min:

From Eq. 7.25, b, we have:

Centre of long cylinder: $\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ ...(7.25, b)
(r = 0)

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 0.428$

Interpolating: $\lambda_1 := 0.8516 + \frac{0.9408 - 0.8516}{10} \cdot 2.8$

i.e. $\lambda_1 = 0.877$

and, $A_1 := 1.0931 + \frac{1.1143 - 1.0931}{10} \cdot 2.8$

i.e. $A_1 = 1.099$

Therefore, $\theta_0 := A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ ...(7.25, b)

i.e. $\theta_0 = 0.489$ *(dimensionless centre tamperature)*

Then, again from Eq. 7.25, b:

$T_0 := T_a + 0.489 \cdot (T_i - T_a)$ *(define centre temperature)*

i.e. $T_0 = 296.7°C$ ***...centre temperature of cylinder after 25 min.***

Note that this value compares well with the value of 297°C obtained by reading Heisler charts.

Surface temperature after 25 min:

From Eq. 7.24, b, we have:

Long cylinder: $\theta(x, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot J_0\left(\frac{\lambda_1 \cdot r}{R}\right)$ ...$Fo > 0.2$ ...(7.24, b)

In Eq. 7.24, b, J_0 is the zeroth order Bessel function of the first kind. Its value can be read from Table 7.2. However, while using Mathcad, J_0 can be got directly by typing '$J_0(\lambda_1)$='.

i.e. $J_0(\lambda_1) = 0.817$

And, while using Mathcad, it is not even necessary to separately obtain the value of $J_0(\lambda_1)$.

See below the expression for T. While calculating the expression for T, value of $J_0(\lambda_1)$ is returned and substituted automatically, and we get the final value of T as shown.

Here, $r/R = 1$, at the surface of the cylinder. So, we get:

$T := (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot J_o(\lambda_1)\right) + T_a$ *(define $T(x, \tau)$)*

i.e. $T = 269.899°C$ ***(temperature at the surface.)***

Compare this with the value of 261.72°C obtained earlier from the charts. The error is in reading the charts.

Amount of heat transferred, Q:

From Eq. 7.27, b we have:

Cylinder: $\frac{Q}{Q_{max}} = 1 - 2 \cdot \theta_0 \cdot \frac{J_1(\lambda_1)}{\lambda_1}$ ...(7.27, b)

i.e. Fraction $:= 1 - 2 \cdot \frac{T_0 - T_a}{T_i - T_a} \cdot \frac{J_1(\lambda_1)}{\lambda_1}$ *(define Fraction, Q/Q_{max})*

i.e. Fraction = 0.556

Now, $Q_{max} = 1.998 \times 10^7$ J *(already calculated)*

Therefore, $Q = Q_{max} \cdot 0.556$ J *(heat transferred in 25 min)*

i.e. $Q = 1.111 \times 10^7$ J ***(heat transferred in 25 min.)***

Note again that this value of Q is quite close to that obtained from Grober's chart.

To draw radial temperature distribution at different times:

Let us draw radial temperature distribution at τ = 15 min, 25 min, 1 hr., etc.

We have, for temperature distribution at any location:

Long cylinder: $$\theta(x, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot J_0\left(\frac{\lambda_1 \cdot r}{R}\right) \quad ...Fo > 0.2 \qquad ...(7.24, b)$$

Centre of long cylnder: $(r = 0)$ $$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \qquad ...(7.25, b)$$

Fourier number as a function of τ: $Fo(\tau) := \frac{\alpha \cdot \tau}{R^2}$ *...for cylinder*

Then, $$T(r, \tau) := \begin{cases} T_a + (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo(\tau)}\right) \text{ if } r = 0 \\ T_a + (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo(\tau)} \cdot J_0\left(\frac{\lambda_1 \cdot r}{R}\right)\right) \text{ otherwise} \end{cases} \qquad ...(A)$$

For a given τ, we will plot Eq. A against r; then, we will repeat for different times, τ.

We use Mathcad to draw the graph. First, define a range variable r, varying from 0 to say, 0.075 m, with an increment of 0.001. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r, 900)$, respectively. Since our aim is to plot for different values of r for given τ, start with τ = 900 s; immediately, this graph is drawn, when we click anywhere outside the graph region. To get the graph for next value of τ =1500, on the y-axis, next to the earlier entry, type a comma and enter $T(r,1500)$ and click anywhere outside the graph region. Repeat this for different values of τ as shown. See Fig. Ex. 7.8.

r := 0, 0.001, ... , 0.075 *(define a range variable r varying from zero to 0.075 m, with an increment of 0.001 m)*

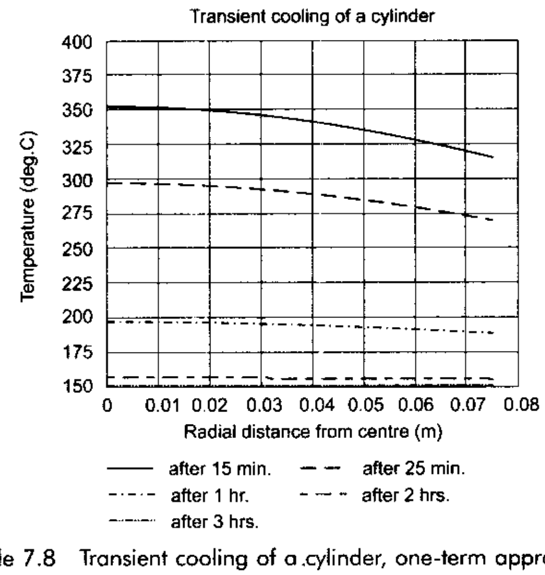

FIGURE Example 7.8 Transient cooling of a .cylinder, one-term approximation solution

Note:

(i) See from the figure how cooling progresses with time. After a time period of 2 hrs the temperatures along the radius are almost uniform, but is yet to reach ambient temperature of 150°C. After about 3 hrs; the body has almost come to equilibrium with the ambient

(ii) Eq. A illustrates a small piece of Mathcad programming. It uses the "if...otherwise" condition, i.e. if $r = 0$, the temperature at the centre is given by Eq. 7.25, b; otherwise, temperature distribution is given by Eq. 7.24, b.

(iii) Observe from the graph that after 25 min, temperature at the centre ($r = 0$) is 296.7°C and at the surface ($r = 0.075$ m), the temperature is 269.9°C as already calculated.

Example 7.9. An apple, which can be considered as a sphere of 8 cm diameter, is initially at a uniform temperature of 25°C. It is put into a freezer at –15°C and the heat transfer coefficient between the surface of the apple and the surroundings in the freezer is 15 W/(m^{2}C). If the thermophysical properties of apple are given to be: $\rho = 840$ kg/m^3, $C_p = 3.6$ kJ/(kgC), $k = 0.513$ W/(mC), and $\alpha = 1.3 \times 10^{-7}$ m^2/s, determine:

(i) centre temperature of the apple after 1 hour,
(ii) surface temperature of apple at that time, and
(iii) amount of heat transferred from the apple.
(iv) draw the temperature profile along the radius for different times.

Solution.

Data:

$R := 0.04$ m $\alpha := 1.3 \cdot 10^{-7}$ m^2/s $k := 0.513$ W/(mC) $C_p := 3600$ J/(kgC) $\rho := 840$ kg/m^3 $T_i := 25$°C

$T_a := -15$°C $h := 15$ W/(m^{2}C) $\tau := 3600$ s

To calculate: the centre temperature after time τ, surface temperature and amount of heat transferred during this period.

First check if lumped system analysis is applicable:

$$Bi = \frac{h \cdot \frac{R}{3}}{k}$$ *(define Biot number...for a sphere, $L_c = (V/A) = R/3$)*

i.e. $Bi = 0.39$ *(Biot number)*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable.**

We will adopt Heisler chart solution and then check the results from one-term approximation solution.

To find the centre temperature after a time period of 3600 s:

For using the charts, now, remember that Bi is defined as:

$$Bi := \frac{h \cdot R}{k}$$ *(define Biot number)*

i.e. $Bi = 1.17$ *(Biot number)*

Fourier number: $$Fo := \frac{\alpha \cdot \tau}{R^2}$$ *(define Fourier number)*

i.e. $Fo = 0.292$ *(Fourier number)*

Also, $$\frac{1}{Bi} = 0.855$$ *(value of 1/Bi.)*

Now, with the value of $Fo = 0.292$, enter the x-axis of Fig. 7.9,a. Move vertically to intersect the $1/B_i = 0.855$ line; from the point of intersection, move horizontally to left, to read on the y-axis $\theta_0 = 0.45$

So, we get:

$$\theta_0 = 0.45 = \frac{T_0 - T_a}{T_i - T_a}$$

i.e. $T_0 := T_a + 0.45 \cdot (T_i - T_a)$ *(define centre temperature)*

i.e. $T_0 = 3$°C ***(centre temperature after 1 hr duration.)***

Surface temperature:

At the surface, $r/R = 1$. Enter Fig. 7.9, b on the x-axis with a value of $1/B_i = 0.855$, move up to intersect the curve of $r/R = 1$, then move to left to read on y-axis the value of $\theta = 0.6$

i.e. $$\theta = \frac{T - T_a}{T_0 - T_a} = 0.6$$

Therefore, $T := 0.6 \cdot (T_0 - T_a) + T_a$ °C *(temperature on the surface)*

i.e. $T = -4.2$°C ***(temperature on the surface.)***

Amount of heat transferred, Q:

We will use Grober's chart, Fig. 7.9, c:

We need $B_i^2 Fo$ to enter the x-axis:

We get $B_i^2 \cdot Fo = 0.4$

With the value of 0.4, enter the x-axis of Fig. 7.9, c, move vertically up to intersect the curve of $B_i = 1.17$, then move horizontally to read $Q/Q_{max} = 0.56$

i.e. from Fig. 7.9, c, we get: $\frac{Q}{Q_{max}} = 0.56$

Now, $Q_{max} = \rho \cdot V \cdot C_p \cdot (T_1 - T_a)$ *(maximum heat transfer possible)*

i.e. $Q_{max} := \rho \cdot \left(\frac{4}{3} \cdot \pi \cdot R^3\right) \cdot C_p \cdot (T_i - T_a)$ *(define Q_{max})*

i.e. $Q_{max} = 3.243 \times 10^4$ J *(maximum heat transfer)*

Therefore, $Q = Q_{max} \times 0.56$ J *(define Q)*

i.e. $Q = 1.816 \times 10^4$ J ***(heat transferred during 1 hr.)***

Verify by one-term approximation solution:

Centre temperature reached after 1 hr:

From Eq. 7.25, c, we have:

Centre of sphere: $(r = 0)$ $\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ ...(7.25, c)

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 1.17$

Interpolating: $\lambda_1 := 1.5708 + \frac{2.0288 - 1.5708}{10} \cdot 1.7$

i.e. $\lambda_1 = 1.649$

and, $A_1 := 1.2732 + \frac{1.4793 - 1.2732}{10} \cdot 1.7$

i.e. $A_1 = 1.308$

Therefore, $\theta_0 := A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ *...from Eq. (7.25, c)*

i.e. $\theta_0 = 0.591$ *...dimensionless centre temperature*

Then, again, from Eq. 7.25, c:

$T_0 := T_a + 0.591 \cdot (T_i - T_a)$ *...define centre temperature*

i.e. $T_0 = 8.64°C$ ***...centre temperature of sphere after 1 hr.***

Compare this value with the value of 3°C obtained by reading the graph; error is due to the error in reading the graph.

Surface temperature after 1 hr:

From Eq. 7.24, c, we have:

Sphere: $\theta(x, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \frac{\sin\left(\frac{\lambda_1 \cdot r}{R}\right)}{\frac{\lambda_1 \cdot r}{R}}$ ...Fo > 0.2 ...(7.24, c)

Here, $r/R = 1$, at the surface of the sphere. So, we get:

$T := (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \frac{\sin(\lambda_1)}{\lambda_1}\right) + T_a$ *...define $T(x, \tau)$*

i.e. $T = -0.711°C$ ***...temperature at the surface.***

Compare this value with the value of −4.2°C obtained by reading the graph; error is due to the error in reading the graph.

Amount of heat transferred, Q:

From Eq. 7.27, c, we have:

Sphere: $\frac{Q}{Q_{max}} = 1 - 3 \cdot \theta_0 \cdot \left(\frac{\sin(\lambda_1) - \lambda_1 \cdot \cos(\lambda_1)}{\lambda_1^3}\right)$ ...(7.27, c) i.e. $\quad \text{Fraction} := 1 - 3\cdot\frac{T_0 - T_a}{T_i - T_a}\cdot\left(\frac{\sin(\lambda_1) - \lambda_1\cdot\cos(\lambda_1)}{\lambda_1^3}\right)$ *(define Fraction, Q/Q_{max})*

i.e. $\quad \text{Fraction} = 0.555$

Now, $\quad Q_{max} = 3.243 \times 10^4$ J *(already calculated)*

Therefore, $\quad Q = Q_{max}\cdot 0.555$ J *(heat transferred in 1 hr)*

i.e. $\quad Q = 1.8 \times 10^4$ J ***(heat transferred in 1 hr.)***

Note again that this value of Q compares well with 1.811×10^4 J, obtained from Grober's chart.

To draw temperature profile along the radius at different times:

We have, for temperature distribution at any location:

$$\theta(x, \tau) = \frac{T(r,\tau) - T_a}{T_i - T_a} = A_1\cdot e^{-\lambda_1^2\cdot Fo}\cdot\frac{\sin\left(\frac{\lambda_1\cdot r}{R}\right)}{\frac{\lambda_1\cdot r}{R}} \quad \text{...Fo} > 0.2 \qquad \text{...(7.24, c)}$$

And, at centre of sphere: $\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1\cdot e^{-\lambda_1^2\cdot Fo}$ ...(7.25, c)

Fourier number as a function of τ: $Fo(\tau) := \frac{\alpha\cdot\tau}{R^2}$ *(for sphere)*

Then,
$$T(r,\tau) := \begin{cases} T_a + (T_1 - T_a)\cdot\left(A_1\cdot e^{-\lambda_1^2\cdot Fo(\tau)}\right) & \text{if } r = 0 \\ T_1 + (T_1 - T_a)\cdot\left[A_1\cdot e^{-\lambda_1^2\cdot Fo(\tau)}\cdot\frac{\sin\left(\frac{\lambda_1\cdot r}{R}\right)}{\frac{\lambda_1\cdot r}{R}}\right] & \text{otherwise} \end{cases} \qquad \text{...(A)}$$

For a given τ, we will plot Eq. A against r; then, we will repeat for different times, τ.

We use Mathcad to draw the graph. First, define a range variable r, varying from 0 to say, 0.04 m, with an increment of 0.001. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and $T(r, 1800)$, respectively. Since our aim is to plot for different values of r for given τ, start with $\tau = 1800$ s; immediately, this graph is drawn, when we click anywhere outside the graph region. To get the graph for next value of $\tau = 3600$, on the y-axis, next to the earlier entry, type a comma and enter $T(r, 3600)$ and click anywhere outside the graph region. Repeat this for different values of t as shown. See Fig. Ex. 7.9.

$r := 0, 0.001, \ldots, 0.04$ *(define a range variable r varying from zero to 0.04 m, with an increment of 0.001 m)*

Note:

(i) See from the Fig. Example 7.9 how cooling progresses with time. After a time period of 5 hrs the temperature along the radius is almost uniform, but is yet to reach ambient temperature of −15°C.

(ii) Eq. A illustrates a small piece of Mathcad programming. It uses the "if...otherwise" condition, i.e. if $r = 0$, the temperature at the centre is given by Eq. 7.25, c; otherwise, temperature distribution is given by Eq. 7.24, c.

Example 7.10. A large concrete slab, one side of which is insulated, is 60 cm thick and is initially at 70°C. The other side is suddenly exposed to hot combustion gases at 1000°C with a heat transfer coefficient of 30 W/(m^{2}C). Determine:

(i) time required for the insulated surface to reach 500°C

(ii) temperature distribution in the wall at that instant

(iii) amount of heat transferred during that time period.

Take average properties of concrete as follows:

$$\rho = 500 \text{ kg/m}^3,\ C_p = 837 \text{ J/(kgC)},\ k = 1.25 \text{ W/(mC)}, \text{ and } a = 0.3 \times 10^{-5} \text{ m}^2\text{/s}.$$

Solution.

Data:

$L := 0.6$ m $\quad \alpha := 0.3\cdot 10^{-5}$ m^2/s $\quad k := 1.25$ W/(mC) $\quad C_p := 837$ J/(kgC) $\quad \rho := 500$ kg/m^3 $\quad T_i := 70$°C

$T_a := 1000$°C $\quad h := 30$ W/(m^{2}C) $\quad T_0 := 500$°C

To calculate: the time τ, at which temperature of insulated surface will reach 500°C, temperature distribution in the slab at that instant, and amount of heat transferred during this period.

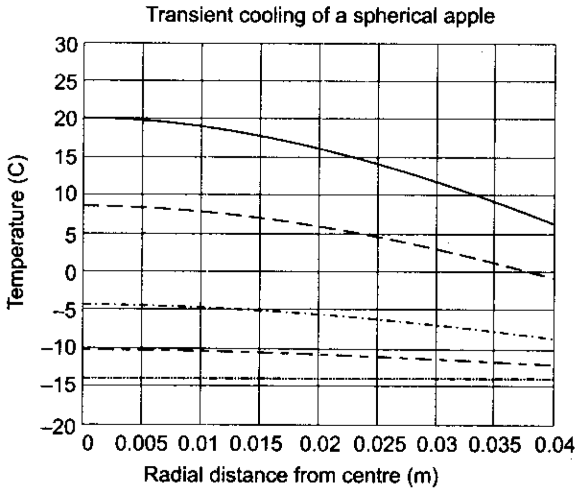

FIGURE Example 7.9 Transient cooling of a sphericl apple, one-term approximation solution

Time required for the insulated surface to reach 500°C:

First of all, recognise that boundary condition at the insulated surface is the same as at the midplane of a slab of half-thickness, L i.e. dT/dx at $x = 0$ is zero.

Therefore, for the present case, we take the thickness of the slab as L.

We will solve this problem by one-term approximation solution:

From Eq. 7.25, a, we have:

Centre of plane wall: $(x = 0)$ $$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \quad ...(7.25, a)$$

Eq. 7.25, a is also **valid for the insulated surface of a wall of thickness** L, as explained above.

$$Bi := \frac{h \cdot L}{k}$$ *(define Biot number)*

i.e. $$Bi = 14.4$$ *(Biot number.)*

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 14.4$

Interpolating: $$\lambda_1 := 1.4289 + \frac{1.4961 - 1.4289}{10} \cdot 4.4$$

i.e. $$\lambda_1 = 1.458$$

and, $$A_1 := 1.2620 + \frac{1.2699 - 1.2620}{10} \cdot 4.4$$

i.e. $$A_1 = 1.265$$

Now, $$\frac{T_0 - T_a}{T_i - T_a} = 0.53763$$

Therefore, Eq. 7.25, a becomes:

$$\frac{0.53763}{1.265} = e^{-1.458^2 \cdot Fo}$$

i.e. $$Fo := \frac{-\ln\left(\frac{0.53763}{1.265}\right)}{(1.458)^2}$$

i.e. $Fo = 0.403$ *(Fourier number)*

Then, $$\tau := \frac{Fo \cdot L^2}{\alpha}\ \text{s}$$ *(define τ, the time require for the insulated surface to reach 500°C)*

i.e. $\tau = 4.8302 \times 10^4$ s ***(time required for the insulated surface to reach 500°C.)***

i.e. $\tau = 13.417$ hrs

To plot the temperature distribution in the slab when τ = 13.417 hrs:

We have to draw temperature as a function of position (i.e. x) for given τ of 13.417 hrs.

We use Eq. 7.24, a, i.e.

Plane wall: $$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\frac{\lambda_1 \cdot x}{L}\right) \quad ...Fo > 0.2 \qquad ...(7.24, a)$$

Therefore, we write:

$$T(x, \tau) := T_a + (T_i - T_a) \cdot \left(A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\frac{\lambda_1 \cdot x}{L}\right)\right)$$

To plot the $T(x, \tau)$ using Mathcad, we first define a range variable x from zero to L = 0.6 m, and then select the x–y graph from the graph palette. On the place holder on the x-axis, fill in x and on the place holder on the y-axis, fill in $T(x, \tau)$. Click anywhere outside the graph region and the graph appears.

$x := 0, 0.01, \ldots, 0.6$ *(range variable x from zero to 0.6 m with an increment of 0.01 m)*

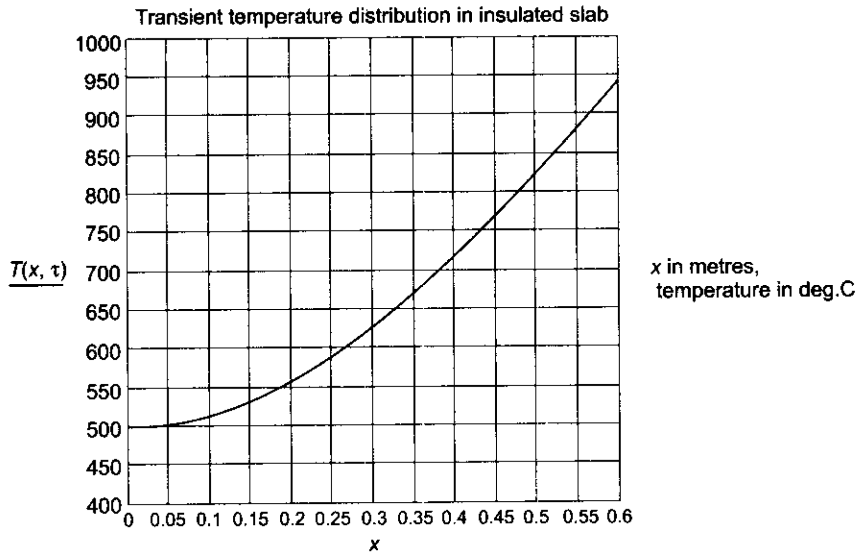

FIGURE Example 7.10 Transient temperature distribution in an insulated slab at the instant τ = 13.417 hrs

Remember: x is measured from the insulated surface.

Amount of heat transferred per unit surface area, Q:

From Eq. 7.27, a, we have:

Plane wall: $$\frac{Q}{Q_{max}} = 1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} \qquad ...(7.27, a)$$

i.e. $$\frac{Q}{Q_{max}} = 1 - \frac{T_0 - T_a}{T_i - T_a} \cdot \frac{\sin(\lambda_1)}{\lambda_1}$$

Now, $Q_{max} := \rho \cdot L \cdot C_p \cdot (T_a - T_i)$ *(Joules per unit surface area)*

i.e. $Q_{max} = 2.335 \times 10^8$ *(Joules per unit surface area)*

Therefore, $$Q := Q_{max} \cdot \left(1 - \frac{T_0 - T_a}{T_i - T_a} \cdot \frac{\sin(\lambda_1)}{\lambda_1}\right) \text{ J/m}^2 \quad \text{...define } Q$$

i.e. $Q = 1.48 \times 10^8$ J/m^2 ***(heat transferred per unit surface area during the process)***

Positive value of Q indicates that heat is transferred into the slab.

7.7 One-dimensional Transient Conduction in Semi-infinite Solids

A solid which has one exposed surface and extends to infinity in other directions is known as a semi-infinite solid. So, change in boundary condition at the exposed surface initiates temperature transients in the solid. One-dimensional transient heat conduction in semi-infinite solids, without heat generation, is of interest because of many practical applications. Common example is that of earth's surface subjected to changes in the ambient conditions, thus causing transient conditions in the soil at some depth from the surface; or, in the case of a thick slab, when the exposed surface is subjected to a temperature variation, in the early stages when the effect is not felt at the distant surface, it can be idealised as a semi-infinite solid, to solve the transient conditions near the surface.

Consider a semi-infinite solid, extending from $x = 0$ to $x = \infty$, initially at a uniform temperature, T_i. There is no internal energy generation. Now, if there is a change in the thermal conditions at the exposed surface at $x = 0$, transient conditions will be induced in the solid. Fig. 7.10 illustrates three possible boundary conditions at the surface:

Case (i): Constant surface temperature:

See Fig. 7.10 (a). The solid is initially at a uniform temperature T_i and for times $\tau > 0$, the boundary surface at $x = 0$ is maintained at temperature T_o. Starting with the differential Eq. for one-dimensional, time dependent conduction, for these boundary conditions, the non-dimensional temperature distribution in the solid is obtained as:

$$\frac{T(x, \tau) - T_0}{T_i - T_0} = \text{erf}\left(\frac{x}{2\sqrt{\alpha \cdot \tau}}\right) \quad \text{...(7.29)}$$

where, $\text{erf}(\zeta)$ is the Gaussian error function defined as:

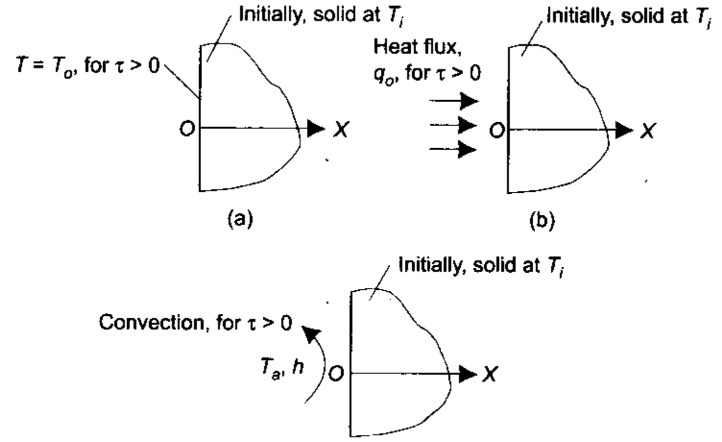

FIGURE 7.10 One-dimensional transient conduction in semi-infinite solids

TABLE 7.3 Values of 'error function'

ζ_1	erf(ζ_1)	ζ_1	erf(ζ_1)
0	0	1	0.8427
0.03	0.0338	1.05	0.8624
0.06	0.0676	1.1	0.8802
0.09	0.1013	1.15	0.8961
0.12	0.1348	1.2	0.9103
0.15	0.168	1.25	0.9229
0.18	0.2009	1.3	0.934
0.21	0.2335	1.35	0.9438
0.24	0.2657	1.4	0.9523
0.27	0.2974	1.45	0.9597
0.3	0.3286	1.5	0.9661
0.33	0.3593	1.55	0.9716
0.36	0.3893	1.6	0.9763
0.39	0.4187	1.65	0.9804
0.42	0.4475	1.7	0.9838
0.45	0.4755	1.75	0.9867
0.48	0.5027	1.8	0.9891
0.51	0.5292	1.85	0.9911
0.54	0.5549	1.9	0.9928
0.57	0.5798	1.95	0.9942
0.6	0.6039	2	0.9953
0.63	0.627	2.05	0.9963
0.66	0.6494	2.1	0.997
0.69	0.6708	2.15	0.9976
0.72	0.6914	2.2	0.9981
0.75	0.7112	2.25	0.9985
0.78	0.73	2.3	0.9989
0.81	0.748	2.35	0.9991
0.84	0.7651	2.4	0.9993
0.87	0.7814	2.45	0.9995
0.9	0.7969	2.5	0.9996
0.93	0.8116	2.55	0.9997
0.96	0.8254	2.6	0.9998
0.99	0.8385	2.65	0.9998
		2.7	0.9999
		2.75	0.9999
		2.8	0.9999

$$\text{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \cdot \int_0^{\zeta} \exp(-u^2)\,du \qquad \text{...(7.30)}$$

Error function is a standard mathematical function. It is integrated numerically and the values are tabulated in Table 7.3.

Gaussian error function is also shown plotted in Fig. 7.11.

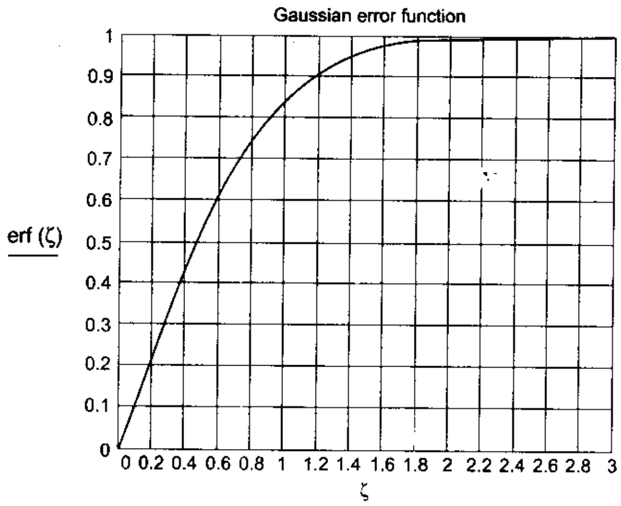

FIGURE 7.11 Gaussian error function erf(ζ) vs. ζ

Remember again that in Fig. 7.11, quantities plotted on x-axis and y-axis are respectively ξ and erf(ξ), with the definition:

$$\zeta = \frac{x}{2\cdot\sqrt{\alpha\cdot\tau}} \quad \text{and,} \quad \text{erf}(\zeta) = \frac{T(x,\tau)-T_0}{T_i-T_0} \qquad \text{from Eq. 7.29}$$

Then, from Eq. 7.29, we have the temperature distribution as:

$$T(x,\tau) = T_0 + (T_i - T_0)\cdot\frac{2}{\sqrt{\pi}}\cdot\int_0^{\frac{x}{\sqrt{4\cdot\alpha\cdot\tau}}} \exp(-u^2)du \qquad \text{...(7.31)}$$

Once the temperature distribution is known, heat flux at any point is obtained by applying Fourier's law, i.e.

$$Q_i = -k\cdot A\cdot\frac{dT(x,\tau)}{dx}\ \text{W}$$ *(instantaneous heat flow rate at a given x location)*

Performing the differentiation on $T(x, \tau)$ given by Eq. 7.31 by Leibnitz's rule, we get,

$$\frac{dT}{dx} = \frac{T_i - T_0}{\sqrt{\pi\cdot\alpha\cdot\tau}}\cdot\exp\left(\frac{-x^2}{4\cdot\alpha\cdot\tau}\right)$$

Substituting this in Fourier's law, we get:

i.e.
$$Q_i = -k\cdot A\cdot(T_i - T_0)\cdot\frac{\exp\left(\frac{-x^2}{4\cdot\alpha\cdot\tau}\right)}{\sqrt{\pi\cdot\alpha\cdot\tau}}\ \text{W} \qquad \text{...(7.32)}$$

Heat flow rate at the surface ($x = 0$):
Putting $x = 0$ in Eq. 7.32,

$$Q_{\text{surface}} = k\cdot A\cdot\frac{(T_0 - T_i)}{\sqrt{\pi\cdot\alpha\cdot\tau}}\ \text{W} \qquad \text{...(7.33)}$$

Total heat flow during $\tau = 0$ to $\tau = \tau$:
This is obtained by integrating Eq. 7.33 from $\tau = 0$ to $\tau = \tau$.

$$Q_{\text{total}} = \frac{k\cdot A\cdot(T_0 - T_i)}{\sqrt{\pi\cdot\alpha}}\cdot\int_0^{\tau}\frac{1}{\tau}d\tau$$

i.e. $$Q_{\text{total}} = k \cdot A \cdot (T_0 - T_i) \cdot 2 \cdot \sqrt{\frac{\tau}{\pi \cdot \alpha}}$$

i.e. $$Q_{\text{total}} = 1.13 \cdot k \cdot A \cdot (T_0 - T_i) \cdot \sqrt{\frac{\tau}{\alpha}} \text{ J} \qquad \text{...(7.34)}$$

Criterion to apply these relations for a finite slab:

For a slab of finite thickness L, above relations for a semi-infinite slab can be applied if:

$$\frac{L}{2 \cdot \sqrt{a \cdot \tau}} \geq 0.5$$

Penetration depth and penetration time:

'Penetration depth' is the distance from the surface where the temperature change is within 1% of the change in the surface temperature, i.e.

$$\frac{T - T_0}{T_i - T_0} = 0.99$$

From Table 7.3, this corresponds to:

$$\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} = 1.8$$

i.e. penetration depth, 'd' is given by:

$$d = 3.6 \cdot \sqrt{\alpha \cdot \tau}$$

'Penetration time' is the time taken for the surface perturbation to be felt at that depth. Therefore,

$$\frac{d}{2 \cdot \sqrt{\alpha \cdot \tau_p}} = 1.8 \quad \text{or,} \quad \tau_p = \frac{d^2}{13 \cdot \alpha}$$

Case (ii): Constant surface heat flux:

See Fig. 7.10 (b). The solid is initially at a uniform temperature T_i, and for times $\tau > 0$, the boundary surface at $x = 0$ is subjected to a constant heat flux q_o.(W/m^2). Then, the temperature distribution in the solid is given as:

$$T(x, \tau) = T_i + \frac{2 \cdot q_0 \cdot \sqrt{\frac{\alpha \cdot \tau}{\pi}}}{k} \cdot \exp\left(\frac{-x^2}{4 \cdot \alpha \cdot \tau}\right) - \frac{q_0 \cdot x}{k} \cdot \left(1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right)\right) \qquad \text{...(7.35)}$$

where, erf is the error function defined ealier.

Case (iii): Convection at the exposed surface:

See Fig. 7.10 (c). The solid is initially at a uniform temperature T_i, and for times $\tau > 0$, the boundary surface at $x = 0$ is subjected to convection with a fluid at temperature T_a and heat transfer coefficient, h.

Then, an energy balance at the surface gives:

$$-k \cdot \left(\frac{dT}{dx}\right)_{x=0} = h \cdot (T_a - T(0, \tau))$$

and, the non-dimensional temperature distribution in the solid is given as:

$$\frac{T(x, \tau) - T_i}{T_a - T_i} = 1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right) - \left(\exp\left(\frac{h \cdot x}{k} + \frac{h^2 \cdot \alpha \cdot \tau}{k^2}\right)\right) \cdot \left(1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} + \frac{h \cdot \sqrt{\alpha \cdot \tau}}{k}\right)\right) \qquad \text{...(7.36)}$$

To represent Eq. 7.36 in graphical form:

Put $$\zeta = \frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} \quad \text{and,} \quad \eta = \frac{h \cdot \sqrt{\alpha \cdot \tau}}{k}$$

then, $\frac{h \cdot x}{k} = 2 \cdot \zeta \cdot \eta$ and, Eq. 7.36 can be written as:

$$\theta(\eta, \zeta) = 1 - \text{erf}(\zeta) - (\exp(2 \cdot \zeta \cdot \eta + \eta^2)) \cdot (1 - \text{erf}(\zeta + \eta)) \quad ...(7.37)$$

Let us plot $\theta(\eta, \zeta)$ against z for various values of η.

We use Mathcad to draw the graph. First, define a range variable ζ, varying from 0 to say, 1.8, with an increment of 0.05. Then, choose x–y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with ζ and $\theta(0.05, \zeta)$, respectively. Since our aim is to plot $\theta(\eta, \zeta)$ for different values of ζ for given η, start with $\eta = 0.05$; immediately, this graph is drawn, when we click anywhere outside the graph region. To get the graph for next value of $\eta = 0.1$, on the y-axis, next to the earlier entry, type a comma and enter $\theta(0.1, \zeta)$, and click anywhere outside the graph region to get the next graph, Repeat this for different values of η as shown.

$\zeta := 0, 0.05, ..., 1.8$ *(define a range variable ζ varying from zero to 1.8, with an increment of* 0.05)

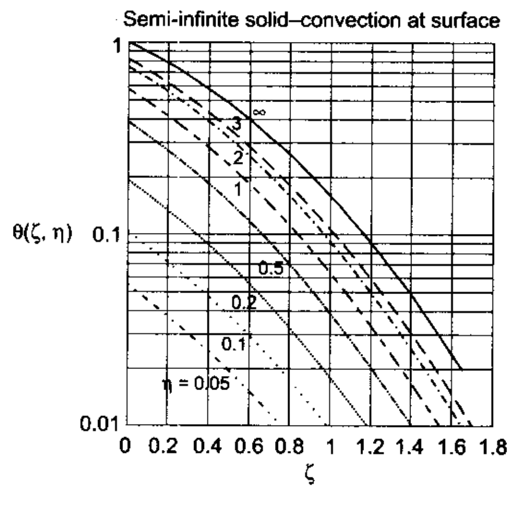

FIGURE 7.12 Non-dimensional temperature for a semi-infinite solid with convection on its surface

Remember again, that in the above graph, definition of ς and η are as follows:

$$\zeta = \frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} \quad \text{and,} \quad \eta = \frac{h \cdot \sqrt{\alpha \cdot \tau}}{k}$$

and, $$\theta(\eta, \zeta) = \frac{T(x, \tau) - T_i}{T_a - T_i}$$

The uppermost curve in the graph is for very large η, and can be taken as for $\eta = \infty$. It signifies $h = \infty$, and this implies that convection resistance is equal to zero and the temperature of the surface is equal to that of the fluid; in other words, this case is equivalent to the case (i) already studied, where surface temperature was suddenly changed to T_o, and then maintained at that temperature for all times $\tau > 0$.

Example 7.11. A thick copper slab ($\alpha = 1.1 \times 10^{-4}$ m^2/s, $k = 380$ W/(mC)) is initially at a uniform temperature of 250°C. Suddenly, its surface temperature is lowered to 60°C.

(i) How long will it take the temperature at a depth of 3 cm to reach 100°C?

(ii) What is the heat flux at the surface at that time?

(iii) What is the total amount of heat removed from the slab per unit surface area till that time?

Solution.

Data:

$\alpha := 1.1 \times 10^{-4}$ m^2/s $\quad k := 380$ W/(mC) $\quad T_i := 250$°C $\quad T_0 := 60$°C $\quad x := 0.03$ m $\quad T := 100$°C

To find the time τ required to reach 100°C at a depth of 0.03 m, surface heat flux and amount of heat transferred during this period.

Time required to reach 100°C at a depth of 3 cm from the surface:

Since this is a very large slab, we will consider it as a semi-infinite medium, with the surface suddenly brought to and maintained at a constant temperature, T_o. This belongs to case (i), refer to Fig. 7.10 (a).

So, Eq. 7.29 is applicable, to get temperature variation as function of position and time, i.e.

$$\frac{T(x,\tau)-T_0}{T_i-T_0} = \text{erf}\left(\frac{x}{2\cdot\sqrt{\alpha\cdot\tau}}\right) \qquad ...(7.29)$$

Now, we get: $\frac{T-T_0}{T_i-T_0} = 0.211$ since all temperatures are given.

From Table 7.3 for values of error function, or from Fig. 7.11, it is seen that:

$$\text{erf}(0.189) = 0.211$$

i.e. $\frac{x}{2\cdot\sqrt{\alpha\cdot\tau}} = 0.189$

Therefore, $\tau := \frac{x^2}{4\cdot 0.189^2\cdot\alpha}$ s *(time required or temperature to reach 100°C at a depth of 3 cm from the surface)*

i.e. $\tau = 57.262$ s ***(time required for temperature to reach 100°C at a depth of 3 cm from the surface.)***

Heat flux at the surface:

This is obtained from Eq. 7.33, i.e.

$$Q_{surface} = k\cdot A\cdot\frac{(T_0-T_i)}{\sqrt{\pi\cdot\alpha\cdot\tau}} \text{ W} \qquad ...(7.33)$$

Therefore, heat flux: $q_{surface} = k\cdot\frac{(T_0-T_i)}{\sqrt{\pi\cdot\alpha\cdot\tau}}$ W/m² *...since $q = Q/A$*

i.e. $q_{surface} = -5.133\times 10^5$ W/m²

Note: negative sign indicates that energy is leaving the surface, which is true, since the slab is being cooled.

Total amount of heat removed, per unit surface area:

This is obtained by integrating Eq. 7.33 from $\tau = 0$ to $\tau = \tau$, and is given by Eq. 7.34, i.e.

$A := 1$ m² *...surface area*

$$Q_{total} := 1.13\cdot k\cdot A\cdot(T_0-T_i)\cdot\sqrt{\frac{\tau}{\alpha}} \text{ J} \qquad ...(7.34)$$

i.e. $Q_{total} = -5.886\times 10^7$ J/m² ***...total heat removed from the slab.***

Note: again, negative sign indicates that heat is leaving the slab.

Example 7.12. A large block of steel ($\alpha = 1.4\times 10^{-5}$ m²/s, $k = 45$ W/(mC)) is initially at a uniform temperature of 25°C. Suddenly, its surface is exposed to a constant heat flux of 3×10^5 W/m². Calculate the temperature at a depth of 3 cm after a period of 1 min.

Solution.

Data:

$\alpha := 1.4\times 10^{-5}$ m²/s $\quad k := 45$ W/(mC) $\quad T_i := 25$°C $\quad q_0 := 3\cdot 10^5$ W/m² $\quad x := 0.03$ m $\quad \tau := 60$ s

To find the temperature after a period of time $\tau = 60$ s, at a depth of 0.03 m.

Temperature at a depth of 3 cm, after a time period of 60 s:

This is the case of a semi-infinite slab, with constant heat flux conditions at its exposed surface. So, this is case (ii), refer Fig. 7.10 (b).

So, Eq. 7.35 is applicable, to get temperature variation as function of position and time, i.e.

$$T(x,\tau) := T_i + \frac{2\cdot q_o\cdot\sqrt{\frac{\alpha\cdot\tau}{\pi}}}{k}\cdot\exp\left(\frac{-x^2}{4\cdot\alpha\cdot\tau}\right) - \frac{q_o\cdot x}{k}\cdot\left(1-\text{erf}\left(\frac{x}{2\cdot\sqrt{\alpha\cdot\tau}}\right)\right) \qquad ...(7.35)$$

Substituting and calculating, we get,

i.e. $T(x, \tau) = 98.949°C$ *(temperature at a depth of 3 cm, after a time period of 60 s.)*

Note: In Mathcad, there is no need to separately find out erf() and substitute, etc. All calculations are done in one step, since error function is one of the built-in functions in Mathcad.

Example 7.13. A thick concrete slab ($\alpha = 7 \times 10^{-7}$ m²/s, $k = 1.37$ W/(mC)) is initially at a uniform temperature of 350°C. Suddenly, its surface is subjected to convective cooling with a heat transfer coefficient $h = 100$ W/(m²C) into an ambient at 30°C. Calculate the temperature 8 cm from the surface, 1 h after the start of cooling.

Solution.

Data:

$\alpha := 7 \cdot 10^{-7}$ m²/s $\quad k := 1.37$ W/(m.C) $\quad T_i := 350°C \quad T_a := 30°C \quad h := 100$ W/(m²C)

$x := 0.08$ m $\quad \tau := 3600$ s

To find the temperature after a period of time $\tau = 3600$ s, at a depth of 0.08 m.

Temperature at a depth of 8 cm, after a time period of 3600 s:

This is the case of a semi-infinite slab, with convection conditions at its exposed surface. So, this is case (iii), refer Fig. 7.10 (c).

So, Eq. 7.36 is applicable, to get temperature variation as function of position and time, i.e.

$$\frac{T(x, \tau) - T_i}{T_a - T_i} = 1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right) - \left(\exp\left(\frac{h \cdot x}{k} + \frac{h^2 \cdot \alpha \cdot \tau}{k^2}\right)\right) \cdot \left(1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} + \frac{h \cdot \sqrt{\alpha \cdot \tau}}{k}\right)\right) \quad ...(7.36)$$

Therefore,

$$T(x, \tau) := T_i + (T_a - T_i) \cdot \left[1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right) - \left(\exp\left(\frac{h \cdot x}{k} + \frac{h^2 \cdot \alpha \cdot \tau}{k^2}\right)\right) \cdot \left(1 - \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} + \frac{h \cdot \sqrt{\alpha \cdot \tau}}{k}\right)\right)\right]$$

i.e. $T(x, \tau) = 287.811°C$ *(temperature at a depth of 8 cm, after a time period of 3600 s.)*

Again, note the ease with which above expression is calculated in Mathcad.

Exercise: Check this result from Fig. 7.12.

To show graphically the progress of cooling at various times:

It is interesting to see how the cooling of the slab progresses with time. So, let us calculate the temperatures reached by the same point, i.e. at a depth of 8 cm from the surface, for different time periods:

$\tau := 0.1, 0.2, ..., 15$ *(define a range variable τ, varying from 0.1 hr to say, 15 hr at an interval of 0.1 hr)*

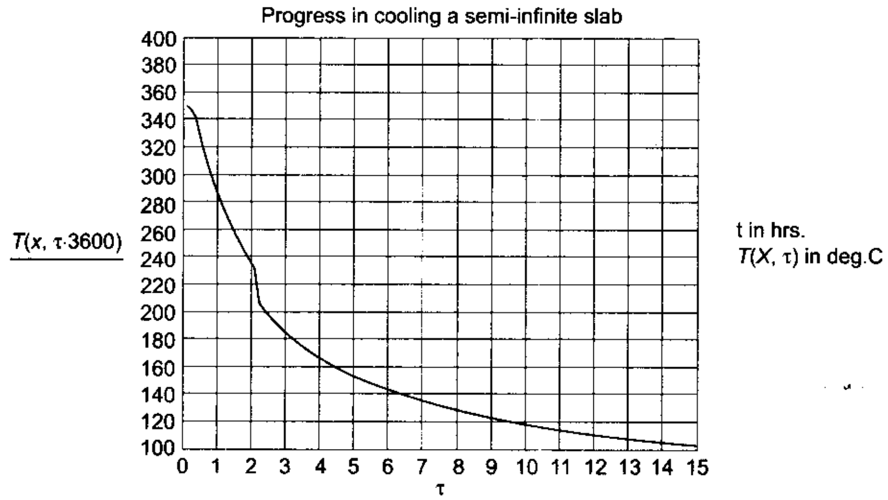

FIGURE Example 7.13 Semi-infinite slab with convection at its surface—Temperature of a point 8 cm below the surface for various time periods, τ

Note: See from the above graph that after about 15 hrs the temperature of the point 8 cm below the surface is approaching a temperature of 100°C.

Example 7.14. In areas where ambient temperature drops to sub zero temperatures and remains so for prolonged periods, freezing of water in underground pipelines is a major concern. It is of interest to know at what depth the water pipes should be buried so that the water does not freeze.

At a particular location, the soil is initially at a uniform temperature of 15°C and the soil is subjected to a sub zero temperature of –20°C continuously for 50 days.

(i) What is the minimum burial depth required to ensure that the water in the pipes does not freeze?, i.e. pipe surface temperature should not fall below 0°C.

(ii) Plot the temperature distributions in the soil for different times i.e. after 1 day, 1 week, etc. Properties of soil may be taken as: $\alpha = 0.138 \times 10^{-6}$ m²/s, $\rho = 2050$ kg/m³, $k = 0.52$ W/(mK), $C_p = 1840$ J/kg.K.

Solution.

Data:

$\alpha := 0.138 \cdot 10^{-6}$ m²/s $\quad k := 0.52$ W/(mC) $\quad T_i := 15°C \quad T_0 := -20°C \quad T := 0.0°C \quad \tau := 50 \cdot 24 \cdot 3600$ s

i.e. $\tau = 4.32 \times 10^6$ s (*time duration of exposure of soil to sub zero temperature*)

To find the depth x required to reach 0°C under these conditions.

Depth at which temperature reaches 0°C:

We shall consider earth's surface as a semi-infinite medium, with the surface suddenly brought to and maintained at a constant temperature, T_o. This belongs to case (i), refer to Fig. 7.10 (a).

So, Eq. 7.29 is applicable, to get temperature variation as function of position and time, i.e.

$$\frac{T(x, \tau) - T_0}{T_i - T_0} = \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right) \qquad ...(7.29)$$

Now, we get: $\quad \dfrac{T - T_0}{T_i - T_0} = 0.571$ since all temperatures are given.

From Table 7.3 for values of error function, or from Fig. 7.11, it is seen that:

$$\text{erf}(0.559) = 0.571$$

i.e. $$\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}} = 0.559$$

Therefore, $\quad x := 0.559 \cdot 2 \cdot \sqrt{\alpha \cdot \tau}$ m $\qquad$ (*define x*)

i.e. $\quad x = 0.863$ m $\qquad$ (***depth at which pipes should be buried to prevent water from freezing.***)

To plot the temperature distributions in the soil at a depth of 1 m for different times, τ:

Again, we use Eq. 7.29. From this equation temperature as a function of x and τ is written as:

$$T(x, \tau) := T_0 + (T_i - T_0) \cdot \text{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right) \qquad ...(A)$$

To plot Eq. A against x for different τ, in Mathcad, first of all define a range variable x varying from 0 to 1 m at an interval of, say, 0.01 m. Then, select x–y graph from the graph palette and fill in the place holder on the x-axis with x and the place holder on the y-axis with $T(x, \tau_1)$, $T(x, \tau_2)$, $T(x, \tau_3)$... etc. where τ_1, τ_2 ... are different times, as desired. Take care that τ is entered in s. Then click anywhere outside the graph region and the graph appears immediately.

$x := 0, 0.01, ..., 1$ $\qquad$ (*define a range variable x, varying from 0 to 1 m, with an increment of 0.01 m*)

Note from the Fig. Example 7.14 that:

(i) even after a period of 50 days of exposure of the surface to an ambient at –20°C, temperature at a depth of 1 m has reached only about 2.5°C.

(ii) after 50 days, freezing temperature of 0°C is reached at a depth of 0.863 m, as calculated.

(iii) slope of the temperature curve, dT/dx, at the surface (i.e. at $x = 0$) decreases as time increases; this means that, heat extracted from the surface decreases as time increases.

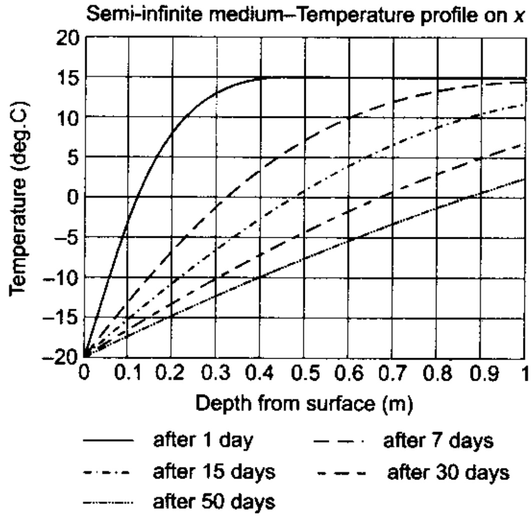

FIGURE Example 7.14 Semi-infinite medium—Temperature variation in 1 m depth for sudden change in surface temperature after different times

7.8 Transient Heat Conduction in Multi-dimensional Systems—Product Solution

In sections 7.6 and 7.7 we considered one-term approximate solutions and Heisler chart solutions for infinite plates, long cylinders, spheres and also for a semi-infinite medium. Underlying assumptions throughout were: one-dimensional conduction and no internal heat generation. However, there are many practical cases where assumption of one-dimensional conduction may not be valid, i.e. temperature gradients may be significant in more than one-dimension. For example, in a 'short cylinder' whose length is comparable to diameter, it is intuitively clear that temperature gradients will be significant in both the longitudinal and radial directions, i.e. the heat transfer will be two-dimensional. Similarly, for a long rectangular bar, it is reasonable to say that heat transfer will be significant in both the x and y directions, and in a parallele piped, heat transfer will be three-dimensional.

7.8.1 Temperature Distribution in Transient Conduction in Multi-dimensional Systems

Some of the common two-dimensional geometries of interest are: a short cylinder, semi-infinite cylinder, infinite rectangular bar, etc. These geometries can be imagined to be obtained by the intersection of any two of the one-dimensional systems studied above and for which one-term approximate solutions or chart solutions are available. Just to give an example, a short cylinder of radius R and length $2L$ can be imagined to be obtained by the intersection of a long cylinder of radius R and an infinite plate of thickness $2L$; an infinite rectangular bar of sides $2L_1$ and $2L_2$ is obtained by the intersection of two infinite plates of thickness $2L_1$ and $2L_2$ respectively, etc.

Now, in such cases, it has been shown (proof is beyond the scope of this book) that for a two-dimensional system, with no internal heat generation, it is possible to construct the solutions for dimensionless temperature distribution in transient heat conduction, by combining the solutions of dimensionless temperature distributions obtained for one-dimensional transient conduction, i.e. the desired two-dimensional solution is given as a product of the one-dimensional solutions of the individual systems which form the two-dimensional body by their intersection. So, in general, we write:

$$\left(\frac{\theta}{\theta_i}\right)_{solid} = \left(\frac{\theta}{\theta_i}\right)_{system\,1} \cdot \left(\frac{\theta}{\theta_i}\right)_{system\,2} \cdot \left(\frac{\theta}{\theta_i}\right)_{system\,3} \quad ...(7.38)$$

LHS of Eq. 7.38 refers to the two or three-dimensional body under consideration and system1, system2 etc. are the one-dimensional systems which by their intersection form the body. (θ/θ_i) is the dimensionless temperature distribution of the one-dimensional system, which is available from Heisler charts or one-term approximation solutions.

Some of the combinations of such one-dimensional systems and the resulting two-dimensional bodies are shown in Fig. 7.13. Remember that for a semi-infinite solid, x coordinate is measured from the surface, and for the plane wall, it is measured from the mid-plane. In Fig.7.13, for convenience, we use the following notations:

$$\theta_{\text{wall}}(x, \tau) = \left(\frac{T(x, \tau) - T_a}{T_i - T_a}\right)_{\text{wall}} = \left(\frac{\theta}{\theta_i}\right)_{\text{wall}} \quad ...(7.39, a)$$

$$\theta_{\text{cyl}}(r, \tau) = \left(\frac{T(r, \tau) - T_a}{T_i - T_a}\right)_{\text{long\_cyl}} = \left(\frac{\theta}{\theta_i}\right)_{\text{long\_cyl}} \quad ...(7.39, b)$$

$$\theta_{\text{semi\_inf}}(x, \tau) = \left(\frac{T(x, \tau) - T_a}{T_i - T_a}\right)_{\text{semi\_inf}} = \left(\frac{\theta}{\theta_i}\right)_{\text{semi\_inf}} \quad ...(7.39, c)$$

With this notation, two-dimensional solution for a long, rectangular bar is given by:

$$\left(\frac{T(x, y, \tau) - T_a}{T_i - T_a}\right)_{\text{rect\_bar}} = \theta_{\text{wall}}(x, \tau) \cdot \theta_{\text{wall}}(y, \tau) \quad ...(7.40)$$

And, two-dimensional solution for a short cylinder is given by:

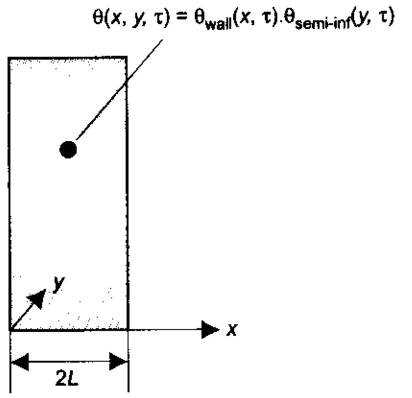

FIGURE 7.13(a) Semi-infinite plate

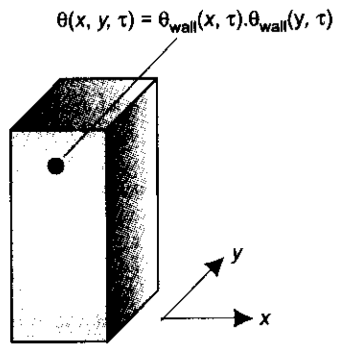

FIGURE 7.13(b) Infinite rectangular bar

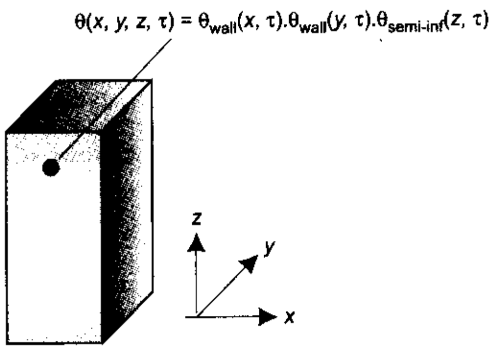

FIGURE 7.13(c) Semi-infinite rectangular bar

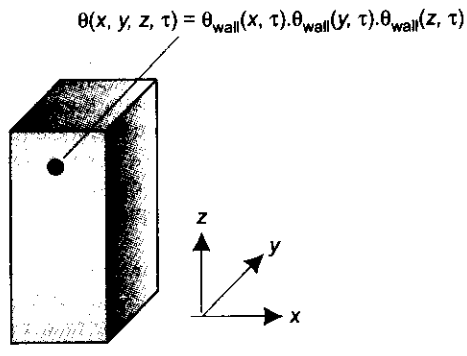

FIGURE 7.13(d) Rectangular parallelepiped

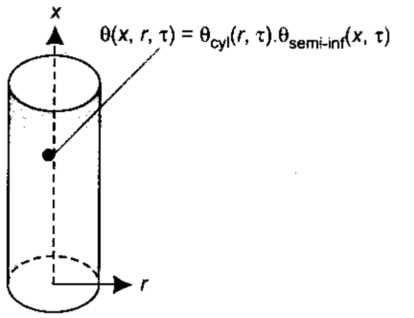

FIGURE 7.13(e) Semi-infinite cylinder

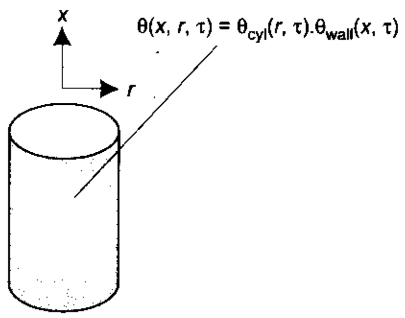

FIGURE 7.13(f) Short cylinder

$$\left(\frac{T(r, x, \tau) - T_a}{T_i - T_a}\right)_{\text{short\_cyl}} = \theta_{\text{wall}}(x, \tau) \cdot \theta_{\text{cyl}}(r, \tau) \quad \text{...(7.41)}$$

Important Note:

(i) Dimensionless temperatures for the one-dimensional systems used to form the product solution for the two/three-dimensional body, must be chosen at the correct locations. In doing so, always remember that for a semi-infinite plate, x is measured from the surface and for an infinite plate, x is measured from the mid-plane.

(ii) If temperature is to be calculated after a given time for the multidimensional body, the solution is straightforward, as shown; however, if the time is to be calculated to attain a given temperature, then, a trial and error solution will be required.

7.8.2 Heat Transfer in Transient Conduction in Multi-dimensional Systems

It has been shown that heat transfer in a multidimensional body in transient conduction can be obtained by using the Grober charts (see Figs. 7.7, 7.8 and 7.9) for Q/Q_{max} for the one-dimensional systems constituting the given multidimensional body.

For a body formed by the intersection of two one-dimensional systems 1 and 2, we have:

$$\left(\frac{Q}{Q_{max}}\right)_{total} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] \quad \text{...(7.42)}$$

For a body formed by the intersection of three one-dimensional systems 1, 2 and 3, we have:

$$\left(\frac{Q}{Q_{max}}\right)_{total} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] + \left(\frac{Q}{Q_{max}}\right)_3 \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_2\right] \quad \text{...(7.43)}$$

Example 7.15. A rectangular aluminium bar 8 cm × 5 cm ($\alpha = 8.4 \times 10^{-5}$ m²/s, $k = 200$ W/(mC), $C_p = 890$ J/(kgC), $\rho = 2700$ kg/m³), is initially at a uniform temperature of $T_i = 200$°C. Suddenly, the surfaces are subjected to convective cooling into an ambient at $T_a = 20$°C, with the convection heat transfer coefficient between the fluid and the surfaces being 300 W/(m²C). Determine the centre temperature of the bar after 1 min from the start of cooling

Solution. Recognise that this is the case of an infinite rectangular bar (Fig. 7.13b), formed by the intersection of two infinite plates, one of thickness $2L_1 = 8$ cm and the other, $2L_2 = 5$ cm.

Therefore, product solution can be adopted to get dimensionless temperature distribution.

Data:

$L_1 := 0.04$ m $\quad L_2 := 0.025$ m $\quad \alpha := 8.4 \cdot 10^{-5}$ m²/s $\quad k := 200$ W/(mC) $\quad \rho := 2700$ kg/m³

$C_p := 890$ J/(kgC) $\quad T_i := 200$°C $\quad T_a := 20$°C $\quad h := 600$ W/(m²C) $\quad \tau := 60$ s

To find: the centre temperature T_0, after time τ, surface temperature and amount of heat transferred

Centre temperature of the slab:

Solution θ is given as the product of the solutions for two infinite slabs 1 and 2:

For slab 1:

$Bi := \frac{h \cdot L_1}{k}$ *(define Biot number)*

i.e. $Bi = 0.12$ *(Biot number.)*

Fourier number: $Fo := \frac{\alpha \cdot \tau}{L_1^2}$

i.e. $Fo = 3.15$

For dimensionless temperature at the centre of the wall, we use Eq. 7.25, a:

Centre of plane wall: $(x = 0)$ $\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ ...(7.25, a)

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 0.12$

Interpolating: $\lambda_1 := 0.3111 + \frac{0.4328 - 0.3111}{10} \cdot 1.2$

i.e. $\lambda_1 = 0.326$

and, $A_1 := 1.0161 + \frac{1.0311 - 1.0161}{10} \cdot 1.2$

i.e. $A_1 = 1.018$

Therefore, $\theta_{01} := A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ *(dimensionless centre temperature for slab 1)*

i.e. $\theta_{01} = 0.729$ *(dimensionless centre temperature for slab 1)*

For slab 2:

$Bi := \frac{h \cdot L_2}{k}$ *(define Biot number)*

i.e. $Bi = 0.075$ *(Biot number)*

Fourier number: $Fo := \frac{\alpha \cdot \tau}{L_2^2}$

i.e. $Fo = 8.064$

For dimensionless temperature at the centre of the wall, we again use Eq. 7.25, a:

A_1 and λ_1 have to be found from Table 7.1, against $B_i = 0.075$

Interpolating: $\lambda_1 := 0.2425 + \frac{0.2791 - 0.2425}{20} \cdot 15$

i.e. $\lambda_1 = 0.27$

and, $A_1 := 1.0098 + \frac{1.013 - 1.0098}{20} \cdot 15$

i.e. $A_1 = 1.012$

Therefore, $\theta_{02} := A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ *(dimensionless centre temperature for slab 2)*

i.e. $\theta_{02} = 0.562$ *(dimensionless centre temperature for slab 2)*

Therefore, dimensionless centre temperature for the two-dimensional slab is given by the product solution:

$\theta_0 := \theta_{01} \cdot \theta_{02}$ *(define θ_0, dimensionless centre temperature for given slab)*

i.e. $\theta_0 = 0.41$ *(demensionless centre temperature for given slab)*

Centre temperature of given slab:

We have: $\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$

Therefore, $T_0 := T_a + \theta_0 \cdot (T_i - T_a)$ *...define centre temperature of slab*

i.e. $T_0 = 93.775°C$ ***(centre temperature of two-dimensional slab.)***

Exercise: Find out the amount of heat transferred per metre length, Q. Also solve this problem, using Heisler and Grober charts. see Fig. 7.7.

Example 7.16. A short, brass cylinder (k = 110 W/(mC), ρ = 8530 kg/m^3, C_p = 389 J/(kgC), and $\alpha = 3.39 \times 10^{-5}$ m^2/s), of 8 cm diameter and height 15 cm is initially at a uniform temperature of T_i = 200°C. The cylinder is placed in a convective environment at 40°C for cooling with an average heat transfer coefficient of 500 W/(m^{2}C).

(i) Determine the temperature at the centre of the cylinder 2 min after the start of the cooling process.

(ii) Determine the centre temperature of the top surface at that time, and

(iii) Determine the heat transfer from the cylinder during this time period.

(iv) Draw the temperature–time history for the centre of the short cylinder

Solution.

Data:

$L := 0.075$ m $\quad R := 0.04$ m $\quad \alpha := 3.39 \times 10^{-5}$ m^2/s $\quad k := 110$ W/(mC) $\quad C_p := 389$ J/(kgC)

$\rho := 8530$ kg/m^3 $\quad T_i := 200$ C $\quad T_a := 40$ C $\quad h := 500$ W/(m^{2}C) $\quad \tau := 120$ s

Recognise that this short cylinder can be considered to be formed by the intersection of a long cylinder of radius R = 4 cm and a plane wall of thickness $2L$ = 15 cm. See Fig. 7.13 (f).

Therefore, product solution can be used. We apply Eq. 7.41, i.e.

$$\left(\frac{T(r, x, \tau) - T_a}{T_i - T_a}\right)_{short\_cyl} = \theta_{wall}(x, \tau) \cdot \theta_{cyl}(r, \tau) \quad ...(7.41)$$

Temperature at the centre of cylinder:

$$\theta(0, 0, \tau) = \theta_{wall}(0, \tau) \cdot \theta_{cyl}(0, \tau)$$

For dimensionless centre temperature of plane wall:

$$Fo := \frac{\alpha \cdot \tau}{L^2} \quad \textit{(Fourier number)}$$

i.e. $\quad Fo = 0.7232 \quad$ *(this is > 0.2)*

$$Bi := \frac{h \cdot L}{k} \quad \textit{(Biot number)}$$

i.e. $\quad Bi = 0.34091 \quad$ *(Biot number)*

For dimensionless temperature at the centre of the wall, we use Eq. 7.25, a:

Centre of plane wall: $$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \quad ...(7.25, a)$$

A_1 and λ_1 have to be found from Table 7.1, against B_i = 0.341

Interpolating: $$\lambda_1 := 0.5218 + \frac{0.5932 - 0.5218}{10} \cdot 4.1$$

i.e. $\quad \lambda_1 = 0.55107$

and, $$A_1 := 1.0450 + \frac{1.0580 - 1.0450}{10} \cdot 4.1$$

i.e. $\quad A_1 = 1.05033$

Therefore, $\quad \theta_{01} := A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \quad$ *(dimensionless centre temperature for slab 1)*

i.e. $\quad \theta_{01} = 0.84323 \quad$ *(dimensionless centre temperature for slab)*

For dimensionless centre temperature of cylinder:

$$Fo := \frac{\alpha \cdot \tau}{R^2} \quad \textit{(Fourier number)}$$

i.e. $\quad Fo = 2.5425 \quad$ *(this is > 0.2)*

$$Bi := \frac{h \cdot R}{k} \quad \textit{(Biot number)}$$

i.e. $\quad Bi = 0.18182 \quad$ *(Biot number)*

A_1 and λ_1 have to be found from Table 7.1, against B_i = 0.182

Intepolating: $$\lambda_1 : = 0.4417 + \frac{0.6170 - 0.4417}{10} \cdot 8.2$$

i.e. $\lambda_1 = 0.58545$

and, $A_1 := 1.0246 + \frac{1.0483 - 1.0246}{10} \cdot 8.2$

i.e. $A_1 = 1.04403$

Therefore, $\theta_{02} := A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$ *(dimensionless centre temperature for clylinder)*

i.e. $\theta_{02} = 0.43677$ *(dimensionless centre temperature for cylinder)*

i.e. $\theta_{cyl}(0, \tau) = 0.43677$

Therefore, $\left(\frac{T(0,0,\tau) - T_a}{T_i - T_a}\right)_{short\_cyl} = (0.84323)\ (0.43677) = 0.3683$

Let $T(0, 0, \tau) = T_{centre}$

i.e. $T_{centre} := T_a + 0.3683 \cdot (T_i - T_a)$ °C ...*temperature at the centre*

i.e. $T_{centre} = 98.928$°C ***(temperature at the centre of short cylinder.)***

Temperature at the centre of top surface of cylinder:

$$\theta(0, L, \tau)_{short\_cyl} = \theta_{wall}(L, \tau) \cdot \theta_{cyl}(0, \tau)$$

Note that centre of top surface of the short cylinder is still at the centre of the long cylinder ($r = 0$) and at the outer surface of intersecting plane wall ($x = L$). First, find the surface temperature of the plane wall: $x = L = 0.075$ m

Now, $\frac{x}{L} = 1$ and, we use Eq. 7.24, a:

Plane wall: $$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\frac{\lambda_1 \cdot x}{L}\right) \quad ...Fo > 0.2 \quad ...(7.24, a)$$

$Fo := 0.723$ $\lambda_1 := 0.55107$ $A_1 := 1.05033$ *(for slab, already calculated.)*

Therefore, $A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos(\lambda_1) = 0.71845$

i.e. $\frac{T(L, \tau) - T_a}{T_i - T_a} = 0.71845$

$$\theta(0, L, \tau)_{short\_cyl} = \theta_{wall}(L, \tau) \cdot \theta_{cyl}(0, \tau)$$

i.e. $\left(\frac{T(0, L, \tau) - T_a}{T_i - T_a}\right)_{short\_cyl} = \theta_{wall}(L, \tau) \cdot \theta_{cyl}(0, \tau) = (0.71845)\ (0.43677) = 0.3138$

Let $T(0, L, \tau) = T_{topsurface\_centre}$

i.e. $T_{topsurface\_centre} := T_a + (T_i - T_a) \cdot 0.3138$

i.e. $T_{topsurface\_centre} = 90.208$°C ***(temperature at the centre of top surface.)***

Heat transfer from the short cylinder:

We use Eq. 7.42:

$$\left(\frac{Q}{Q_{max}}\right)_{total} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right] \quad ...(7.42)$$

First, determine Q_{max}:

$Q_{max} = \rho \cdot (\pi \cdot R^2 \cdot 2 \cdot L) \cdot C_p \cdot (T_i - T_a)$ J *(maximum heat transfer = $m \cdot C_p \cdot \Delta T$)*

i.e. $Q_{max} = 4.00295 \times 10^5$ J *(maximum heat transfer)*

Now, dimensionless heat transfer ratio Q/Q_{max} is determined for both the geometries from eqns. (7.27), or from Grober's charts, i.e. Figs. 7.7 (c) and 7.8 (c).

For the plane wall:

$B_i := 0.341$ $F_o := 0.723$ *(already calculated)*

$\lambda_1 := 0.55107$ $\theta_0 := 0.84323$

Plane wall: $$\frac{Q}{Q_{max}} = 1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} \quad ...(7.27, a)$$

Therefore, $1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} = 0.19881$

i.e. $\left(\frac{Q}{Q_{max}}\right)_1 = 0.19881$

For the long cylinder:

$B_i := 0.18182 \qquad Fo := 2.5425$ *(already calculated)*

$\lambda_1 := 0.58545 \qquad \theta_0 := 0.43677$

Cylinder: $$\frac{Q}{Q_{max}} = 1 - 2 \cdot \theta_0 \cdot \frac{J_1(\lambda_1)}{\lambda_1}$$ ...(7.27, b)

Therefore, $1 - 2 \cdot \theta_0 \cdot \frac{J_1(\lambda_1)}{\lambda_1} = 0.58168$

i.e. $\left(\frac{Q}{Q_{max}}\right)_2 = 0.58168$

Now, apply Eq. 7.42:

$$\left(\frac{Q}{Q_{max}}\right)_{total} = \left(\frac{Q}{Q_{max}}\right)_1 + \left(\frac{Q}{Q_{max}}\right)_2 \cdot \left[1 - \left(\frac{Q}{Q_{max}}\right)_1\right]$$

i.e. $\left(\frac{Q}{Q_{max}}\right)_{total} = 0.19881 + 0.58168\ (1 - 0.19881)$

i.e. $\left(\frac{Q}{Q_{max}}\right)_{total} = 0.66485$

i.e. $Q := Q_{max} \cdot 0.66485\ J$ *(define Q)*

i.e. $Q = 2.66136 \times 10^5\ J$ ***(heat transferred from the short cylinder during the time period of 120 s.)***

Exercise: Work out this problem using the Heisler charts & Grober's charts.

To draw temperature–time history for centre of short cylinder:

Let us rewrite the values of λ_1 and A_1 for wall and cylinder as follows:

For infinite wall:

$B_i := 0.34091$

Therefore: $\lambda_{wall} := 0.55107$

and, $A_{wall} := 1.05033$

For infinite cylinder

$Bi := 0.18182$

Therefore: $\lambda_{cyl} := 0.58545$

and, $A_{cyl} := 1.04403$

Fourier number of wall as a function of τ: $Fo_{wall}(\tau) := \frac{\alpha \cdot \tau}{L^2}$

Fourier number of cylinder as a function of τ: $Fo_{cyl}(\tau) := \frac{\alpha \cdot \tau}{R^2}$

We have for dimensionless centre temperatures:

Centre of plane wall: $(x = 0)$ $$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$$ ...(7.25, a)

Centre of long cylinder: $(r = 0)$ $$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$$ ...(7.25, b)

Then, centre temperature of the short cylinder is given as a function of time as follows:

$$T_{centre}(\tau) := \left| \begin{array}{l} \theta_{c\_wall} \leftarrow A_{wall} \cdot e^{-\lambda_{wall}^2 \cdot Fo_{wall}(\tau)} \\ \theta_{c\_cyl} \leftarrow A_{cyl} \cdot e^{-\lambda_{cyl}^2 \cdot Fo_{cyl}(\tau)} \\ \theta_{centre} \leftarrow \theta_{c\_wall} \cdot \theta_{c\_cyl} \\ T_a + (T_i - T_a) \cdot \theta_{centre} \end{array} \right. \quad \text{...(A)}$$

Therefore, $T_{centre}(120) = 98.92705$ ***(checks with earlier result)***

$\tau := 0, 10, \ldots, 1000$ *(define a range variable τ, varying from 0 to 1000 s, with an increment of 10 s)*

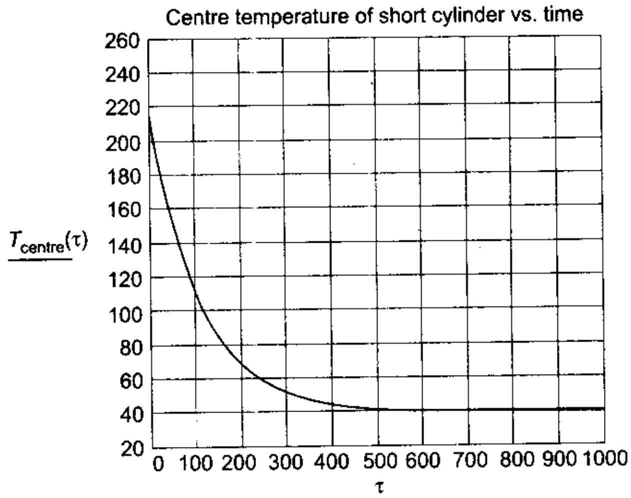

FIGURE Example 7.16 Temperature–time history for centre temperature of a short cylinder

Note:

(i) Note from the graph that centre temperature reaches the ambient temperature after about 600 s.

(ii) Eq. A is a piece of Mathcad programming. LHS defines the function $T(\tau)$; on the RHS, there are 4 lines. First line defines dimensionless centre temperature of infinite wall, next line defines dimensionless centre temperature of long cylinder; third line defines dimensionless centre temperature of short cylinder and the last line defines the temperature at the centre of short cylinder.

(iii) By defining Fourier number as a function of τ, we ensure that for each new τ, new values of Fo are calculated for the wall as well as the cylinder.

(iv) Above graph is important, particularly when the problem is to find the time required for the centre of the short cylinder to reach a given temperature. Then, construct the above graph and then read off the value of time against the desired temperature. For example, from the graph, we see that time required for the centre temperature to reach 85°C is about 150 s.

(v) We can also use the solve block to find accurately the time required for the centre temperature to reach 85°C, as shown below.

(vi) In the above graph, a $\tau = 0$, centre temperature is shown as 215.5°C and not 200°C; this error is due to the fact that two one-term approximation solutions are multiplied together.

$\tau := 100$ s *(trial value of τ)*

Given

$T_{centre}(\tau) = 85$

$\text{Find}(\tau) = 149.65588$ s ***...time required for the centre temperature to reach 85°C.***

Interpolation with Mathcad:

In all the above examples, A_1 and λ_1 for given B_i were found out by manual interpolation from Table 7.1. However, this interpolation can be done easily and accurately in Mathcad, as follows: First, prepare Table 7.1 as an ASCII file, with the name :Coeff.prn. Then, read this file into a matrix M by the command READPRN, as follows:

M := READPRN("Coeff.prn")

Then, extract the columns of this matrix to get Biot number and values of $\lambda_1$and A_1 for plane wall, cylinder and sphere. Remember that columns of the matrix are generally numbered starting from zero. Matrix M has 7 columns: 0, 1,...6. 0th column gives a vector of Biot numbers, 1st column gives λ_1 values for wall, 2nd column gives A_1 values for wall, 3rd and 4th columns give λ_1 and A_1 values for cylinder and, 5th and 6th columns give λ_1 and A_1 values for sphere, respectively.

$$\text{Biot} := M^{\langle 0\rangle} \quad \lambda\text{1wall} := M^{\langle 1\rangle} \quad \text{A1wall} := M^{\langle 2\rangle} \quad \lambda\text{1cyl} := M^{\langle 3\rangle}$$
$$\text{A1cyl} := M^{\langle 4\rangle} \quad \lambda\text{1sph} := M^{\langle 5\rangle} \quad \text{A1sph} := M^{\langle 6\rangle}$$

Then, use the 'linterp' function for linear interpolation. Here, each column must have the same number of values. If there are two vectors X and Y giving a series of x and y values, for any given x-value, y-value is obtained by: linterp(X, Y, x-value). This command performs the linear interpolation to give the y-value corresponding to desired x-value.

Let us define functions to quickly get λ_1 and A_1 for wall, cylinder and sphere, for given Biot number:

λ_1_Wall(Bi) := linterp(Biot, λ_1 wall, Bi) *(defines λ_1 for wall, for given Bi)*

example: λ_1_wall(0.341) = 0.55107

A_1_wall(Bi) := linterp(Biot, A_1 wall, Bi) *(defines A_1 for wall, for given Bi)*

example: A_1_wall(0.341) = 1.05033

λ_1_cyl (Bi) := linterp(Biot, λ_1 cyl, Bi) *(defines λ_1 for cylinder, for given Bi)*

example: λ_1_cyl(0.18182) = 0.58513

A_1_cyl(Bi) := linterp(Biot, A_1 cyl, Bi) *(defines A_1 for cylinder, for given Bi)*

example: A_1_cyl(0.18182) = 1.04399

λ_1_sph(Bi) := linterp(Biot, λ_1 sph, Bi) *(defines λ_1 for sphere, for given Bi)*

example: λ_1_sph(0.25) = 0.84005

A_1_sph(Bi) := linterp(Biot, A_1 sph, Bi) *(defines A_1 for sphere, for given Bi.)*

example: A_1_sph(0.25) = 1.0736.

Compact Mathcad program to find the centre temperature of short cylinder:

Above problem can be solved in a single step by the following Mathcad program:

$$T_{centre}(L, R, h, k, T_i, T_a, \tau, \alpha) = \left| \begin{array}{l} Bi\_\text{wall} \leftarrow \dfrac{h \cdot L}{k} \\ Bi\_\text{cyl} \leftarrow \dfrac{h \cdot R}{k} \\ \lambda_{\text{wall}} \leftarrow \lambda_1\_\text{wall}(Bi\_\text{wall}) \\ A_{\text{wall}} \leftarrow A_1\_\text{wall}(Bi\_\text{wall}) \\ \lambda_{\text{cyl}} \leftarrow \lambda_1\_\text{cyl}(Bi\_\text{cyl}) \\ A_{\text{cyl}} \leftarrow A_1\_\text{cyl}(Bi\_\text{cyl}) \\ Fo_{\text{wall}} \leftarrow \dfrac{\alpha \cdot \tau}{L^2} \\ Fo_{\text{cyl}} \leftarrow \dfrac{\alpha \cdot \tau}{R^2} \\ \theta_{\text{c\_wall}} \leftarrow A_{\text{wall}} \cdot e^{-\lambda_{\text{wall}}^2 \cdot Fo_{\text{wall}}} \\ \theta_{\text{c\_cyl}} \leftarrow A_{\text{cyl}} \cdot e^{-\lambda_{\text{cyl}}^2 \cdot Fo_{\text{cyl}}} \\ \theta_{\text{centre}} \leftarrow \theta_{\text{c\_wall}} \cdot \theta_{\text{c\_cyl}} \\ T_a + (T_i - T_a) \cdot \theta_{\text{centre}} \end{array} \right.$$

LHS of the above program defines the centre temperature of the short cylinder as a function of the variables L, R, h, k, T_i, T_a, τ and α. RHS has 12 lines. First two lines define the Biot number for wall and cylinder, respectively. In 3rd and 4th lines, we get the λ_1 and A_1 for the wall using the interpolation functions defined earlier. In 5th and 6th lines λ_1 and A_1 are calculated for the long cylinder. In 7th and 8th lines, Fourier numbers are calculated for wall and cylinder, respectively. Centre temperatures of wall and long cylinder are calculated in lines 9 and 10, respectively. In 11th line, dimensionless centre temperature of short cylinder is calculated as a product solution. Finally, the last line gives the temperature at the centre of the short cylinder.

Advantage of this program is that it is quick and gives accurate calculation of the final result, i.e. the centre temperature of the short cylinder. However, the disadvantage is that values calculated in the intermediate steps are not available outside the program.

For the above problem:

$L := 0.075$ m $R := 0.04$ m $\alpha := 3.39 \cdot 10^{-5}$ m^2/s $k := 110$ W/(mC) $T_i := 200°C$ $T_a := 40°C$

$h := 5000$ W/(m^{2}C) $\tau := 120$ s

Therefore, we get:

$T_{centre}(L, R, h, k, T_i, T_a, \tau, \alpha) = 98.984°C$ *(centre temperature of short cylinder.)*

Compare this with the value of 98.93°C obtained earlier. Difference is due to the truncation errors crept in to the solution in the earlier case.

7.9 Summary of Basic Equations

Basic relations derived in this chapter are summarised below in Table 7.4, for convenience and ready reference.

TABLE 7.4 Basic relations for transient conduction

Relation	Comments
$\frac{d^2T}{dx^2} = \frac{1}{\alpha} \cdot \frac{dT}{d\tau}$	Governing differential equation in Cartesian cords. for one-dimensional, transient cond. without heat generation.
$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right)$ if $Bi < 0.1$	Lumped system analysis, $Bi = \frac{h \cdot L_c}{k}$ and $L_c = \frac{V}{A}$
$\frac{\theta}{\theta_i} = \frac{T(\tau) - T_a}{T_i - T_a} = \exp(-Bi \cdot Fo)$ if $B_i < 0.1$	$Fo = \frac{\alpha \cdot \tau}{L_c^2}$ = **Fourier number, or relative time**
$\frac{\rho \cdot C_p \cdot V}{h \cdot A} = t$	Time constant (seconds)
$Q(\tau) = m \cdot C_p \cdot \frac{dT(\tau)}{d\tau}$, W $Q(\tau) = h \cdot A \cdot (T(\tau) - T_a)$, W	Instantaneous heat transfer rate
$Q_{tot} = m \cdot C_p \cdot (T(\tau) - T_i)$, J $Q_{tot} = \int_0^\tau Q(\tau) d\tau$, J	Total heat transfer from time = 0 to τ
$Q_{max} = m \cdot C_p \cdot (T_a - T_i)$, J	Maximum heat transfer
$\frac{T(\tau) - T_a}{T_i - T_a} = \exp(-a \cdot \tau) + \frac{\frac{b}{a}}{T_i - T_a} \cdot (1 - \exp(-a \cdot \tau))$ $a = \frac{h \cdot A}{\rho \cdot V \cdot C_p}$ $b = \frac{q \cdot A}{\rho \cdot V \cdot C_p}$	Temperature distribution when transient condition is induced by mixed B.C. (e.g. a slab with constant heat flux, q, at one surface and convection at the other surface)
$\tau = -\frac{1}{a} \cdot \ln\left[\frac{T(\tau) - T_a - \left(\frac{b}{a}\right)}{T_i - T_a - \left(\frac{b}{a}\right)}\right]$	Time required to attain a given temperature in the above case
$T(\tau) = T_a + \frac{b}{a} = T_a + \frac{q}{h}$	Steady state temperature for the above case (obtained by putting $\tau = \infty$, in Eq. 7.20

Contd.

Contd.

$\theta(x, \tau) = \dfrac{T(x, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \cos\left(\dfrac{\lambda_1 \cdot x}{L}\right)$ ...$Fo > 0.2$	One-term approximation solution for plane wall
$\theta(x, \tau) = \dfrac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot J_0\left(\dfrac{\lambda_1 \cdot r}{R}\right)$ ...$Fo > 02.$	One-term approximation solution for long cylinder
$\theta(x, \tau) = \dfrac{T(r, \tau) - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo} \cdot \dfrac{\sin\left(\dfrac{\lambda_1 \cdot r}{R}\right)}{\dfrac{\lambda_1 \cdot r}{R}}$ ...$Fo > 0.2$	One-term approximation solution for a sphere
$\theta_0 = \dfrac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$	One-term approximation-centre temperature for plane wall
$\theta_0 = \dfrac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$	One-term approximation-centre temperature for long cylinder
$\theta_0 = \dfrac{T_0 - T_a}{T_i - T_a} = A_1 \cdot e^{-\lambda_1^2 \cdot Fo}$	One-term approximation-centre temperature for sphere
$\dfrac{Q}{Q_{max}} = 1 - \theta_0 \cdot \dfrac{\sin(\lambda_1)}{\lambda_1}$	Dimensionless heat transfer for large, plane wall
$\dfrac{Q}{Q_{max}} = 1 - 2 \cdot \theta_0 \cdot \dfrac{J_1(\lambda_1)}{\lambda_1}$	Dimensionless heat transfer for long cylinder
$\dfrac{Q}{Q_{max}} = 1 - 3 \cdot \theta_0 \cdot \left(\dfrac{\sin(\lambda_1) - \lambda_1 \cdot \cos(\lambda_1)}{\lambda_1^3}\right)$	Dimensionless heat transfer for a sphere
Semi-infinite slab:	
$\dfrac{T(x, \tau) - T_0}{T_i - T_0} = erf\left(\dfrac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right)$	Dimensionless temperature distribution in a semi-infinite slab, surface temperature suddenly changed to T_0
$T(x, \tau) = T_0 + (T_1 - T_0) \cdot \dfrac{2}{\sqrt{\pi}} \cdot \int_0^{\frac{x}{\sqrt{4 \cdot \alpha \cdot \tau}}} \exp(-u^2) du$	Temperature distribution in a semi-infinite slab, surface temperature suddenly changed to T_0
$Q_{surface} = k \cdot A \cdot \dfrac{(T_0 - T_i)}{\sqrt{\pi \cdot \alpha \cdot \tau}}$, W	Heat flow rate at the surface, for above case
$Q_{total} = 1.13 \cdot k \cdot A \cdot (T_0 - T_i) \cdot \sqrt{\dfrac{\tau}{\alpha}}$, J	Total heat flow during time period τ for the above case

Semi-infinite slab:

Temperature distribution in a semi-infinite slab, surface is subjected to constant heat flux, q_0:

$$T(x, \tau) = T_i + \frac{2 \cdot q_0 \cdot \sqrt{\dfrac{\alpha \cdot \tau}{\pi}}}{k} \cdot \exp\left(\frac{-x^2}{4 \cdot \alpha \cdot \tau}\right) - \frac{q_0 \cdot x}{k} \cdot \left(1 - \mathrm{erf}\left(\frac{x}{2 \cdot \sqrt{\alpha \cdot \tau}}\right)\right)$$

Contd.

Semi-infinite slab:
Temperature distribution in a semi-infinite slab, surface is subjected to convection at its surface:
$\frac{T(x,\tau)-T_i}{T_a-T_i} = 1-\operatorname{erf}\left(\frac{x}{2\cdot\sqrt{\alpha\cdot\tau}}\right)-\left(\exp\left(\frac{h\cdot x}{k}+\frac{h^2\alpha\cdot\tau}{k^2}\right)\right)\cdot\left(1-\operatorname{erf}\left(\frac{x}{2\cdot\sqrt{\alpha\cdot\tau}}+\frac{h\cdot\sqrt{\alpha\cdot\tau}}{k}\right)\right)$
Multidimensional transient conduction:
Temperature distribution for a body formed by intersection of three bodies:
$\left(\frac{\theta}{\theta_i}\right)_{solid}=\left(\frac{\theta}{\theta_i}\right)_{system1}\cdot\left(\frac{\theta}{\theta_i}\right)_{system2}\cdot\left(\frac{\theta}{\theta_i}\right)_{system3}$
Temperature distribution in long, rectangular bar:
$\left(\frac{T(x,y,\tau)-T_a}{T_i-T_a}\right)_{rect\_bar}=\theta_{wall}(x,\tau)\cdot\theta_{wall}(y,\tau)$
Temperature distribution in short cylinder:
$\left(\frac{T(x,y,\tau)-T_a}{T_i-T_a}\right)_{short\_cyl}=\theta_{wall}(x,\tau)\cdot\theta_{cyl}(r,\tau)$
Heat transfer in two-dimensional transient conduction:
$\left(\frac{Q}{Q_{max}}\right)_{total}=\left(\frac{Q}{Q_{max}}\right)_1+\left(\frac{Q}{Q_{max}}\right)_2\cdot\left[1-\left(\frac{Q}{Q_{max}}\right)_1\right]$
Heat transfer in three-dimensional transient conduction:
$\left(\frac{Q}{Q_{max}}\right)_{total}=\left(\frac{Q}{Q_{max}}\right)_1+\left(\frac{Q}{Q_{max}}\right)_2\cdot\left[1-\left(\frac{Q}{Q_{max}}\right)_1\right]+\left(\frac{Q}{Q_{max}}\right)_3\cdot\left[1-\left(\frac{Q}{Q_{max}}\right)_1\right]\cdot\left[1-\left(\frac{Q}{Q_{max}}\right)_2\right]$

7.10 Summary

In this chapter, we dealt with transient conduction, i.e. time dependent conduction, for three important, simple geometries, namely, plane slab, long cylinder and sphere. In general, in transient conduction, temperature within the body depends both on time and spatial coordinates. However, when the resistance for conduction within the body is negligible as compared to the convective resistance at the surface of the body, analysis becomes simpler and we adopt 'lumped system analysis', i.e. the whole body heats up or cools down as a 'lump', and the temperature within the body is uniform, and is a function of time only. This is characterised by the value of non-dimensional Biot number (B_i) being less than 0.1. When Biot number is more than 0.1, results for temperature distribution become more complicated and are obtained as infinite series. However, if the non-dimensional time, Fourier number (Fo) is more than 0.2, it is found that considering only the first term of the infinite series and neglecting rest of the terms, introduces an error of no more than 2%. Such an approximate solution is known as 'one-term approximation'. Coefficients for use in the one-term approximation have been tabulated. Now, the same results are presented in graphical form too, known as 'Heisler charts' for all the three geometries considered. However, these graphs are subject to reading errors and, whenever better accuracy is desired, relations for one-term approximation should be used.

Dimensionless heat transfer during transient conduction may be obtained either from one- term approximaion solutions, or from the 'Grober's charts', also given for the three geometries.

'Product solution' was explained for multidimensional transient conduction, when the temperature variation in a given body cannot be considered as one-dimensional, if the body in question could be considered as having been formed by the intersection of two or more one-dimensional systems for which solutions are available.

Just as in the case of steady state conduction, in transient conduction too analytical methods have their limitation, i.e. difficulty in taking into account complex shape of the body, varying boundary conditions, or accounting for varying thermophysical properties and heat transfer coefficients. In such cases, numerical methods should be preferred since it is simple to handle such problems with numerical methods.

In the next chapter, we shall study numerical methods, as applied to steady state and transient conduction.

Questions

1. Differentiate between transient conduction and steady state conduction.
2. What do you understand by 'lumped system analysis'? What are the underlying assumptions? What is the criterion to apply lumped system analysis?
3. Explain the importance and physical significance of: Biot number and Fourier number, in transient conduction.
4. In which situation is lumped system analysis likely to be applicable—in water or in air? Why?
5. With usual notations, show that temperature distribution in a body during Newtonian heating or cooling is given by:

$$\frac{T(\tau)-T_a}{T_i-T_a} = \exp\left(\frac{-h \cdot A \cdot \tau}{\rho \cdot C_p \cdot V}\right) \qquad \text{...[V.T.U.]}$$

6. For transient conduction with negligible internal resistance, prove that:

$$\frac{\theta}{\theta_i} = \frac{T(\tau)-T_a}{T_i-T_a} = \exp(-Bi \cdot Fo) \qquad \text{...[M.U.]}$$

7. Discuss the effect of Biot number and Fourier number on 'time constant' of a thermocouple. ...[M.U.]
8. What are Heisler charts? Explain their significance in solving transient conduction problems. ...[V.T.U.]
9. What is meant by 'one-term approximation solution'? When is it applicable?
10. What is the use of Grober's charts?
11. What do you mean by a 'semi-infinite medium'? In what situations the assumption of semi-infinite medium appropriate?
12. Explain the 'product solution method' for multidimensional transient conduction problems. What is the main precaution to be taken while using this method?

Problems

Lumped system analysis:

1. A large copper slab, 5 cm thick at a uniform temperature of 350°C, suddenly has its surface temperature lowered to 30°C. Find the time at which the slab temperature becomes 100°C. Given: $\rho = 9000$ kg/m^3, $c_p = 0.38$ kJ/(kgK), $k = 370$ W/(mK), $h = 100$ W/(m^{2}K). Also, find out the rate of cooling after 60 seconds.
2. An aluminium plate ($\rho = 2707$ kg/m^3, $C_p = 0.896$ kJ/(kgC), and $k = 200$ W/(mC)) of thickness 3 cm is at an initial, uniform temperature of 40°C. Suddenly, it is subjected to uniform heat flux $q = 7000$ W/m^2, on one surface while the other surface is exposed to an air stream at 20°C, with a heat transfer coefficient of $h = 60$ W/(m^{2}C).
   (i) Is lumped system analysis applicable to this case?
   (ii) If yes, plot the temperature of the plate as a function of time, and
   (iii) What is the temperature of the plate in steady state?
3. A household electric iron has an aluminium base ($\rho = 2700$ kg/m^3, $C_p = 0.896$ kJ/(kgC), and $k = 200$ W/(mC)), which weighs 1.4 kg. Total area of iron is 0.05 m^2 and is heated with a 500 W heating element. Initially, the iron is at ambient temperature of 20°C. How long will it take for the iron to reach 120°C once it is switched on? Take heat transfer coefficient between iron and the ambient air as 18 W/(m^{2}K).
4. A copper ball of 8 cm diameter, initially at a uniform temperature of 350°C is suddenly placed in an environment at 90°C. Heat transfer coefficient h, between the ball and the fluid is 100 W/(m^{2}K). For copper, c_p = 0.383 kJ/(kgK), $\rho = 8954$ kg/m^3, $k = 386$ W/(mK). Calculate the time required for the ball to reach a temperature of 150°C. Also, find the rate of cooling after 1 hr. Show graphically how the temperature of the sphere falls with time.
5. A 12 mm diameter, mild steel sphere initially at a uniform temperature of 540°C is suddenly placed in an air stream at 27°C, with a heat transfer coefficient h of 114 W/(m^{2}C). For mild steel, $c_p = 0.475$ kJ/(kgK), $\rho = 7850$ kg/m^3, $k = 42.5$ W/(mK), $\alpha = 0.043$ m^2/hr.

(i) Calculate the time required for the ball to reach a temperature of 95°C.

(ii) Also, find the instantaneous heat transfer rate two minutes after the commencement of cooling....[V.T.U.]

6. A steel bar of diameter 6 cm is to be annealed by cooling it slowly from 850°C to 150°C in an ambient at 30°C. Heat transfer coefficient between the surface of the bar and the ambient is 40 W/(m^{2}C). Determine the time required for annealing. For steel, c_p = 0.5 kJ/(kgK), ρ = 7800 kg/m^3, k = 50 W/(mK).

7. An egg with a mean diameter of 40 mm and initially at 20°C is placed in boiling water for 4 min and found to be boiled to the consumer's taste. For how long should a similar egg for the same consumer be boiled when taken from a refrigerator at 5°C? Take the following properties for the egg: c_p = 2.0 kJ/(kgK), ρ = 1200 kg/m^3, k = 10 W/(mK).

   Take value of heat transfer coefficient h = 100 W/(m^{2}C). ...[M.U.]

8. A thermocouple junction is in the form of 4 mm diameter sphere. Properties of the material are C_p = 420 J/(kgK), ρ = 8000 kg/m^3, k = 40W/(mK), This junction, initially at 40°C, is inserted in a stream of hot air at 300°C, with h = 45W/(m^{2}K). Find:

   (i) time constant of the thermocouple.

   (ii) thermocouple is taken out from hot air after 10 sec and is kept in still air at 30°C

   Assuming heat transfer coefficient in air as 10 W/(m^{2}K), find the temperature attained by the junction 20 sec after removing from hot air stream. ...[M.U.]

9. A thermocouple junction is in the form of 3 mm diameter sphere. Properties of the material are: C_p= 400 J/(kgK), r = 8600 kg/m^3, k = 30W/(mK). This junction, is inserted in a gas stream to measure temperature, with a heat transfer coefficient of h = 45 W/(m^{2}K). How long will it take for the thermocouple to record 98% of the applied temperature difference?

One-term approximate solution and Heisler charts:

10. A large plate of aluminium 5 m thick, is initially at 250°C, and it is exposed to convection with a fluid at 75°C with a heat transfer coefficient of 500 W/(m^{2}K). Calculate the temperature at a depth of 1.25 cm from one of the faces, one minute after the plate is exposed to the fluid. What is the amount of heat removed from the plate during this time?

    Take themophysical properties of aluminium as: c_p = 0.9 kJ/(kgK), ρ = 2700 kg/m^3, k = 215 W/(mK), α = 8.4 × 10^{-5} m^2/s.

11. A steel plate (α = 1.2 × 10^{-5} m^2/s, k = 43 W/(mC)), of thickness $2L$ = 8 cm, initially at a uniform temperature of 200°C is suddenly immersed in an oil bath at T_a = 40°C. Convection heat transfer coefficient between the fluid and the surface is 700 W/(m^{2}C). How long will it take for the centre plane to cool to 90°C? What fraction of the energy is removed during this time?

12. A long, 15 cm diameter cylindrical shaft made of stainless steel 304 (k = 14.9 W/(mC), ρ = 7900 kg/m^3, C_p = 477 J/(kgC), and α = 3.95 × 10^{-6} m^2/s), is initially at a temperature of 250°C. The shaft is then allowed to cool slowly in an ambient at 40°C, with an average heat transfer coefficient of 85 W/(m^{2}C).

    (i) Determine the temperature at the centre of the shaft 15 min after the start of the cooling process.

    (ii) Determine the surface temperature at that time, and

    (iii) Determine the heat lost per unit length of the shaft during this time period.

13. A solid brass sphere (k = 60 W/(mC), α = 1.8 × 10^{-5} m^2/s) of 18 cm diameter is initially at 150°C. It is cooled in an environmemt at 20°C with a heat transfer coefficient of 600 W/(m^{2}C).

    (i) How long will it take for the centre of the sphere to reach 50°C?

    (ii) Also, calculate the fraction of energy removed from the sphere during this time.

    (iii) Draw the radial temperature profile after different time durations at intervals of 15 min.

14. A heavily insulated steel pipe line is 1 m in diameter and is 40 mm thick. Initially, the wall is at a uniform temperature of −15°C. Suddenly, a hot fluid at 75°C enters the pipe with a heat transfer coefficient of 600 W/(m^{2}C) between the fluid and the inner surface.

    (i) Calculate the temperature on outer metal surface 10 min after the hot fluid is let in to the pipe.

    (ii) What is the heat flux from the fluid to the pipe at that time?, and

    (iii) How much energy is transferred per metre length of pipe during this time interval?

    [*Hint:* Since diameter >> thickness of pipe, the pipe wall may be considered as a plane slab. This is a plane slab of thickness L, insulated at one surface; therefore, its insulated surface is equivalent to the mid-plane of a plane slab of thickness $2L$. (See Example 7.10) Find B_i and Fo, and apply the one-term approximation solution formulas for temperature distibution and heat transferred. Heat flux at the inner surface is obtained by first calculating the temperature T_i at the inner surface (i.e. at x/L = 1), and then, by Newton's equation i.e. $q = h(T_i - T_a)$. You may also check your results by Heisler and Grober charts.]

having been formed by the intersection of two or more one-dimensional systems for which solutions are available.

Just as in the case of steady state conduction, in transien... limitation, i.e. difficulty in taking into accou... accounting for varying th... method...

Semi-infinite medium:

15 A thick aluminium slab, ($\alpha = 8.4 \times 10^{-5}$ m^2/s, k = 200 W/(mC)) initially at 250°C, has its surface temperature suddenly lowered to and maintained 40°C.

(i) How long will it take the temperature at a depth of 4 cm to reach 100°C?

(ii) What is the heat flux at the surface at that time?

(iii) What is the total amount of heat removed from the slab per unit surface area till that time?

16. A thick concrete slab, ($\alpha = 7 \times 10^{-7}$ m^2/s, k = 1.37 W/(mC)) initially at 350°C, has its surface suddenly exposed to a convection environment at 30°C, with a heat transfer coefficient of 100 W/(m^{2}C). What is the temperature at a depth of 8 cm from the surface after a period of 1 hour?

17. A large block of steel ($\alpha = 1.4 \times 10^{-5}$ m^2/s, k = 45 W/(mC)) is initially at a uniform temperature of 20°C. Suddenly, its surface is exposed to a constant heat flux of 3.5×10^5 W/m^2. Calculate the temperature at a depth of 4 cm after a period of 2 min.

18. In areas where ambient temperature drops to sub-zero temperatures and remains so for prolonged periods, freezing of water in underground pipelines is a major concern. It is of interest to know at what depth the water pipes should be buried so that the water does not freeze.
At a particular location, the soil is initially at a uniform temperature of 15°C and the soil is subjected to a sub-zero temperature of – 15°C continuously for 60 days.

(i) What is the minimum burial depth required to ensure that the water in the pipes does not freeze? (i.e. pipe surface temperature should not fall below 0°C.)

(ii) Plot the temperature distributions in the soil for different times i.e. after 1 day, 1 week, etc.

Properties of soil may be taken as: $\alpha = 0.138 \times 10^{-6}$ m^2/s, ρ = 2050 kg/m^3, k = 0.52 W/(mK), C_p = 1840 J/kgK.

19. A motor car weighing 1350 kg is moving at a speed, u = 50 km/h. It is stopped in 5 sec by 4 brakes with brake bands of 250 cm^2 area each, pressing against steel drums. Assuming that the brake lining and the drum surfaces are at the same temperature and that the heat is dissipated by flowing across the surface of the drums (assumed t be very thick), find the maximum temperature rise.
[*Hint*: K.E. of the vehicle, $\{(1/2)m.u^2\}$ is dissipated in a time of t = 5 sec. i.e. heat flow rate $Q = \{(\text{K.E.})/t\}$ is known. Then, considering the drum surface as semi-infinite slab, apply Eq. 7.33 to get $(T_o - T_i)$].

Product solution:

20. A rectangular aluminium bar 6 cm × 3 cm ($\alpha = 8.4 \times 10^{-5}$ m^2/s, k = 200 W/(mC), C_p = 890 J/(kgC), ρ = 2700 kg/m^3), is initially at a uniform temperature of T_i = 150°C. Suddenly the surfaces are subjected to convective cooling into an ambient at T_a = 20°C, with a convection heat transfer coefficient between the fluid and the surfaces being 250 W/(m^{2}C).

(i) Determine the centre temperature of the bar after 1 min from the start of cooling

(ii) What is the heat transferred per metre length of the bar during this period?

21. A short aluminium cylinder (k = 200 W/(mC), ρ = 2700 kg/m^3, C_p = 890 J/(kgC), and $\alpha = 8.4 \times 10^{-5}$ m^2/s), of 8 cm diameter and height 4 cm is initially at a uniform temperature of T_i = 200°C. The cylinder is subjected to convective cooling with a fluid at 20°C, with an average heat transfer coefficient of 300 W/(m^{2}C).

(i) Determine the temperature at the centre of the cylinder 1 min after the start of the cooling process.

(ii) Determine the centre temperature of the top surface at that time, and

(iii) Determine the heat transfer from the cylinder during this time period.

22. A 20 cm long, 15 cm diameter aluminium block ($\alpha = 9.75 \times 10^{-5}$ m^2/s, k = 236 W/(mC), C_p = 896 J/(kgC), ρ = 2700 kg/m^3), is initially at a uniform temperature of 25°C. The block is heated in a furnace at 1100°C till the centre temperature reaches 250°C. If the heat transfer coefficient on all surfaces of the block is 60 W/(m^{2}C), determine how long the block should remain in the furnace.
[*Hint*: This short cylinder is considered as obtained by the intersection of an infinite plate and an infinite cylinder. Solution involves trial and error method: For a range of times, calculate the centre temperature of the short cylinder and plot a graph of time vs. centre temperature. From this graph, read the time corresponding to a centre temperature of 250°C. While selecting the time range, be careful to see that the desired centre temperature of 250°C is bracketed by the results obtained for the time range.]

23. A solid lead cylinder 0.5 m in diameter and 0.5 m in length, initially at a uniform temperature of 150°C, is dropped into a medium at 20°C in which the heat transfer coefficient is 1200 W/(m^{2}C). Plot the temperature–time history of the centre of this cylinder.

Appendix

Mathcad functions for Transient conduction for Sab, Cylinder and Sphere ...One term approximation ($Fo > 0.2$):

1. Plane wall:

Values of λ_1:

$$\lambda_1 := 1.5 \qquad \textit{(guess value)}$$

Given

$$\lambda_1 \cdot \tan(\lambda_1) = Bi$$

$$\lambda_{1\,wall}(Bi) := \text{Find}(\lambda_1) \qquad \text{((A7.1)...}\textit{Function to determine } \lambda_1 \textit{ as a function of } Bi\text{)}$$

$$\lambda_{1\,wall}(40) = 1.5325 \qquad \textit{(Example)}$$

Values of A_1:

$$A_{1wall}(Bi) := \frac{4 \cdot (\lambda_{1\,wall}(Bi))}{2 \cdot \lambda_{1\,wall}(Bi) + \sin(2 \cdot \lambda_{1\,wall}(Bi))} \qquad \text{((A7.2)...}\textit{Function to determine } A_1\text{)}$$

$$A_{1\,wall}(100) = 1.2731 \qquad \textit{(Example)}$$

Centre temp. of plane wall:

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a} \qquad (T_0 = \text{centre temp., } T_i = \text{initial temp., } T_a = \text{ambient temp.})$$

$$\theta_{0wall}(Bi, Fo) := A_{1wall}(Bi) \cdot \exp(-\lambda_{1wall}(Bi)^2 \cdot Fo) \qquad \text{((A7.3)...}\textit{Function to determine centre temp, of plane wall}\text{)}$$

$$\theta_{0wall}(1, 3) = 0.121 \qquad \textit{(Example)}$$

Temp. at any location in a plane wall:

$$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} \qquad xbyL = \frac{x}{L}$$

$$\theta_{wall}(Bi, Fo, xbyL) := A_{1wall}(Bi) \cdot \exp(-\lambda_{1wall}(Bi)^2 \cdot Fo) \cdot \cos(\lambda_{1wall}(Bi) \cdot xbyL)$$

((A7.4)...*Function to determine temp. at any location in plane wall*)

$$\theta_{wall}(1, 3, 0) = 0.121 \qquad \textit{(Example)}$$

Heat transfer in a plane wall:

$$\frac{Q}{Q_{max}} = 1 - \theta_0 \cdot \frac{\sin(\lambda_1)}{\lambda_1} \qquad \text{...where } Q_{max} = mCp(T_a - T_i)$$

$$Q_{by}Q_{max\,wall}(Bi, Fo) := 1 - \theta_{0\,wall}(B_i, Fo) \cdot \frac{\sin(\lambda_{1\,wall}(Bi))}{\lambda_{1\,wall}(Bi)}$$

Example A7.1. An Aluminium slab 10 cm thick, is initially at an uniform temperature of 600°C. It is suddenly immersed in a liquid at 90°C and heat is transferred with a heat transfer coeff. of 1100 W/(m^2.K). Determine;

(i) temperature at the centre line after 1 min.

(ii) temperature at the surface after 1 min.

(iii) total energy removed per unit area of the slab during this time period

Thermophysical data for Aluminium are: $\alpha = 8.85 \times 10^{-5}$ m^2/s, k = 215 W/(m.K), ρ = 2700 kg/m^3, C_p = 900 J/(kg.K)

Solution.

Date:

$L := 0.05$ (heat thickness) $\quad \alpha := 8.85 \cdot 10^{-5}$ m^2/s $\quad k := 215$ W/(m.C) $\quad \rho := 2700$ kg/m^3

$C_p := 900$ J/(kg.K) $\quad T_i := 600$°C $\quad T_a := 90$°C $\quad h := 1100$ W(m^2.C) $\quad t := 60$ s

To calculate: the centre line temp., surface temp. and energy transferred per unit surface area of slab.

First check if lumped system analysis is applicable:

$$Bi := \frac{h \cdot L}{k} \qquad \textit{...define Biot number}$$

i.e. $\quad Bi = 0.256 \qquad$ *...Biot number.*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable.** We will adopt one-term approximation solution.

To find the centre line temp.:

Fourier number: $Fo := \frac{\alpha \cdot \tau}{L^2}$ i.e. $Fo = 2.124$ (> 0.2...*therefore, one term approx. is applicable*)

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$$ (T_0 = *centre temp.*, T_i = *initial temp.*, T_a = *ambient temp.*)

$$\theta_{0\,wall}(Bi, Fo) := A_{1\,wall}(Bi) \cdot \exp(-\lambda_{1\,wall}(Bi)^2 \cdot Fo)$$ (*Function to determine centre temp. of plane wall*)

Therefore,

$$\theta_{0wall}(Bi, Fo) = 0.63$$

And, $T_0 := 0.63 \cdot (T_i - T_a) + T_a$

i.e. $T_0 = 411.3°C$ (*Centre line temp....***Ans.**)

Surface temperature:

At the surface, $x/L = 1.$

$$\theta(x, \tau) = \frac{T(x, \tau) - T_a}{T_i - T_a} \quad xbyL = \frac{x}{L}$$

We have:

$$\theta_{wall}(Bi, Fo, xbyL) := A_{1wall}(Bi) \cdot \exp(\lambda_{1wall}(Bi)^2 \cdot Fo) \cdot \cos(\lambda_{1wall}(Bi) \cdot xbyL$$ (*Function to determine temp. at any location in plane wall*)

Therefore,

$$\theta_{wall}(Bi, Fo, 1) = 0.557$$ (*at the surface, since* $x/L = 1$)

And, $T := 0.557 \cdot (T_i - T_a) + T_a$

i.e. $T = 374.07°C$ (*Surface temp...***Ans.**)

Amount of heat transferred, Q, in one minute:

$$\frac{Q_{max}}{A} = \frac{\rho \cdot C_p \cdot V \cdot (T_i - T_a)}{A} = \rho \cdot C_p \cdot (2 \cdot L) \cdot (T_i - T_a)\ J/m^2$$ (*max. heat trans. per unit area*)

We have:

$$\theta_{by}Q_{max\,wall}(Bi, Fo) := 1 - \theta_{wall}(Bi, Fo) \cdot \frac{\sin(\lambda_{1\,wall}(Bi))}{\lambda_{1\,wall}(Bi)}$$ (*Function to determine* Q/Q_{max})

i.e. $Q_{by}Q_{max\,wall}(Bi, Fo) = 0.394$

Therefore, $Q_{by}A := 0.394 \cdot [\rho \cdot C_p \cdot (2 \cdot L) \cdot (T_i - T_a)]$

i.e. $Q_{by}A = 4.883 \cdot 10^7\ J/m^2$ (*heart tr. per unit area from the slab in one min...***Ans.**)

2. Infinite Cylinder:

Values of λ_1:

$$\lambda_1 := 1.5$$ (*guess value*)

Given

$$\lambda_1 \cdot \frac{J_1(\lambda_1)}{J_0(\lambda_1)} = Bi$$

$$\lambda_{1cyl}(Bi) := \text{Find}(\lambda_1)$$ ((A7.6)...*Function to determine* λ_1 *as a function of Bi*)

$$\lambda_{1cyl}(10) = 2.1795$$ (*Example*)

Values of A_1:

$$A_{1cyl}(Bi) := \frac{2 \cdot J_1(\lambda_{1\,cyl}(Bi))}{\lambda_{cyl}(Bi) \cdot (J_0(\lambda_{1\,cyl}(Bi))^2 + J_1(\lambda_{1\,cyl}(Bi)^2)}$$ ((A7.7)...*Function to determine* A_1)

$$A_{1cyl}(10) = 1.5677$$ (*Example*)

Centre temp. of long cylinder:

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$$ (T_0 = *centre temp.*, T_i = *initial temp.*, T_a = *ambient temp.*)

$$\theta_{0cyl}(Bi, Fo) := A_{1cyl}(Bi) \cdot \exp(-\lambda_{1\,wall}(Bi)^2 \cdot Fo)$$ ((A7.8)...*Function to determine centre temp. of long cylinder*)

$$\theta_{0cyl}(0.1, 18) = 0.031$$ (*Example*)

Temp. at any radius in a long cylinder:

$$\theta(r,\ \tau) = \frac{T(r,\ \tau) - T_a}{T_i - T_a}$$ *(r = any radius, r_0 = outeer radius of cyl.)*

$$r_{by}r_0 := \frac{r}{r_0}$$

$$\theta_{cyl}(Bi, Fo, r_{by}r_0) := A_{1cyl}(Bi)\cdot \exp(-\lambda_{1cyl}(Bi)^2\cdot Fo)\cdot J_0(\lambda_{1cyl}(Bi)\cdot r_{by}r_0)$$

$\theta_{cyl}(0.1, 18, 0) = 0.031$ *(Example...(A7.9)...Function to determine temp. at any location)*

Heat transfer in a long cylinder:

$$\frac{Q}{Q_{max}} = 1 - 2\cdot \theta_0 \cdot \frac{J_1(\lambda_1)}{\lambda_1}$$ *(where $Q_{max} = mCp(T_a - T_i)$)*

$$Q_{by}Q_{max\ cyl}(Bi, Fo) := 1 - 2\cdot \theta_{0cyl}(Bi, Fo)\cdot \frac{J_1(\lambda_{1cyl}(Bi))}{\lambda_{1cyl}(Bi)}$$ *((A7.10)...Function to determine Q/Q_{max})*

$Q_{by}Q_{max\ cyl}(1, 1) = 0.797$ *(Example)*

Example A7.2. A long stainless steel shaft 10 cm in diameter is initially at an uniform temperature of 25°C. It is placed in a furnace at 950°C and the heat transfer coeff. is 150 W/(m^2.K).

(i) Calculate the time required for the axis temperature to reach 700 C
(ii) what is the temperature at a radial position of 3 cm from the centre at that time?
(iii) what is the amount of heat transferred per unit length during this time period?

For steel, $\alpha = 3.954 \times 10^{-6}$ m2/s, k = 14.9 W/(m.C), ρ = 7900 kg/m^3, C_p = 477 J/(kg.C)

Solution.

Data:

$L := 1$ m $r_0 := 0.05$ m $\alpha := 3.954\cdot 10^{-6}$ m^2/s $k := 14.9$ W/(m.C) $C_p := 477$ J/(kg.C) $Ti := 25$°C

$T_a := 950$°C $h := 150$ W/(m^2.C) $T_0 := 700$°C *(axis temp.)*

To calculate: the time τ, temp. at a rad. of 3 cm, and amount of heat transferred during this period.

First check if lumped system analysis is applicable:

$$Bi := \frac{h\cdot \frac{r_0}{2}}{k}$$ *(define Biot number...for a cylinder, $L_c = (V/A) = r_0/2$)*

i.e. $Bi = 0.252$ *(Biot number.)*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable.** We will adopt one term approximation solution.

To find the time reqd. for the centre line temp. to reach 700°C:

For one term approximation, now remember that Bi is defined as:

$$Bi := \frac{h\cdot r_0}{k}$$ *(define Biot number)*

i.e. $Bi = 0.503$ *(Biot number)*

Fourier number: $$Fo = \frac{\alpha\cdot \tau}{r_0^2}$$ *(define Fourier number)*

We have:

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$$ *(T_0 = centre temp., T_i = initial temp., T_a = ambient temp.)*

$$\theta_{0cyl}(Bi, Fo) := A_{1cyl}(Bi)\cdot \exp(-\lambda_{1cyl}(Bi^2)\cdot Fo)$$ *(Function to determine centre temp. of long cylinder)*

i.e. $$\frac{T_0 - T_a}{T_i - T_a} = 0.27027$$

i.e. the function θ_{0cyl} is equal to 0.27027. Let us calculate the fourier no. to satisfy this requirement. We use the Solve block of Mathcad:

$F_o := 0.2$ *(guess values)*

Given

$$\theta_{0cyl}(Bi, Fo) = 0.27027$$

$$\text{Find}(Fo) = 1.592$$

i.e. Fourier number: $F_o := 1.592$

Note: Observe the ease with which above calculation is perfomed with Mathcad.

i.e. $\tau := \dfrac{Fo \cdot r_0^2}{\alpha}$

i.e. $\tau = 1.007 \cdot 10^3$ s *(time reqd. for the centre line to reach 700°C...***Ans.***)*

Temperature at a radial distance of 3 cm from centre:

At the required posotion, $r/r_0 = 3/5.$

i.e. $r_{by}r_0 := 0.6$

We have:

$$\theta_{0cyl}(Bi, Fo, r_{by}r_0) := A_{1cyl}(Bi) \cdot \exp(\lambda_{1cyl}(Bi)^2 \cdot Fo) \cdot J_0(\lambda_{1cyl}(Bi) \cdot r_{by}r_0)$$

(Function to determine temp. at any location)

i.e. $\theta_{cyl}(Bi, Fo, r_{by}r_0) = 0.249$

And,

$$\theta(r, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a}$$

(r = any radius, r_0 = outer radius of cyl.)

i.e. $T := 0.249 \cdot (T_i - T_a) + T_a$

i.e. $T = 719.675$°C *(temp. at radial distance of 3 cm...***Ans.***)*

Amount of heat transferred, Q:

Now, $Q_{max} = \rho \cdot V \cdot C_p \cdot (T_a - T_i)$ *(max. heat transfer possible)*

i.e. $Q_{max} := \rho(\pi \cdot r_0^2 \cdot L) \cdot C_p \cdot (T_a - T_i)$ *(define Q_{max})*

i.e. $Q_{max} = 2.738 \cdot 10^7$ J *(max. heat transfer)*

We have:

$$Q_{by}Q_{max\,cyl}(Bi, Fo) := 1 - 2 \cdot \theta_{0cyl}(Bi, Fo) \cdot \frac{J_1(\lambda_{1cyl}(Bi))}{\lambda_{1cyl}(Bi)}$$

(Function to determine Q/Q_{max})

i.e. $Q_{by}Q_{max\,cyl}(Bi, Fo) = 0.759$

Therefore, $Q := Q_{max} \cdot 0.759$

i.e. $Q = 2.078 \cdot 10^7$ J *(amount of heat transferred when the centre line reached 700°C, i.e. in 1007 seconds...***Ans.***)*

3. Spheres

Values of λ_1:

$\lambda_1 := 2.5$ *(guess value)*

Given

$$1 - \lambda_1 \cdot \cot(\lambda_1) = Bi$$

$\lambda_{1sph}(Bi) := \text{Find}(\lambda_1)$ ((A7.11)...*Function to determine λ_1 as a function of Bi)*

$\lambda_{1sph}(10) = 2.8363$ *(Example)*

Values of A_1:

$$A_{1sph}(Bi) := \frac{4 \cdot (\sin(\lambda_{1sph}(Bi)) - \lambda_{1sph}(Bi) \cdot \cos(\lambda_{1sph}(Bi)))}{2 \cdot \lambda_{1sph}(Bi) - \sin(2 \cdot \lambda_{1sph}(Bi))}$$

((A7.120)...*Function to determine A_1)*

$A_{1sph}(100) = 1.999$ *(Example)*

Centre temp. of Sphere:

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$$

(T_0 = centre temp., T_i = initial temp., T_a = ambient temp.)

$$\theta_{0sph}(Bi, Fo) := A_{1sph}(Bi) \cdot \exp(-\lambda_{1sph}(Bi)^2 \cdot Fo)$$

((A7.13)...*Function to determine centre temp. of sphere)*

$\theta_{0sph}(0.02, 30) = 0.167$ *(Example)*

Temp. at any location in a sphere:

$$\theta(r, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a}$$ *(r = any radius, r_0 = outer radius of cyl.)*

$$r_{by}r_0 := \frac{r}{r_0}$$

$$\theta_{sph}(Bi, Fo, r_{by}r_0) := \begin{Vmatrix} A_{1\,sph}(Bi)\cdot\exp(-\lambda_{1\,sph}(Bi)^2\cdot Fo \text{ if } r_{by}r_0 = 0 \\ A_{1\,sph}(Bi)\cdot\exp(-\lambda_{1\,sph}(Bi)^2\cdot Fo)\cdot\dfrac{\sin(\lambda_{1\,sph}(Bi)\cdot r_{by}r_0)}{(\lambda_{1\,sph}(Bi)\cdot r_{by}r_0)} \text{ otherwise} \end{Vmatrix}$$

((A7.14)...*Function to determine temp. at any location)*

$\theta_{sph}(0.02, 30, 0) = 0.167$ *(Example)*

Heat transfer in a sphere:

$$\frac{Q}{Q_{max}} = 1 - 3\theta_0\cdot\left(\frac{\sin(\lambda_1) - \lambda_1\cdot\cos(\lambda_1)}{\lambda_1^3}\right)$$ *(where $Q_{max} = mC_p(T_a - T_i)$)*

$$Q_{by}Q_{max\,sph}(Bi, Fo) := 1 - 3\cdot\theta_{0sph}(Bi, Fo)\cdot\left(\frac{\sin(\lambda_{1sph}(Bi)) - \lambda_{1sph}(Bi)\cdot\cos(\lambda_{1sph}(Bi))}{\lambda_{1\,sph}(Bi)^3}\right)$$

((A7.15)...*Function to determine Q/Q_{max})*

$Q_{by}Q_{max\,sph}(1, 1) = 0.916$ *(Example)*

Example A7.4. A stainless steel sphere, 10 mm in diameter is initially at an uniform temperature of 450°C. It is suddenly placed in a water bath at 25°C and the heat transfer coeff. is 6000 W/(m^2.K).

(i) Calculate the time required for the centre temperature to reach 50 C

(ii) what is the temperature at the surface of the sphere at that time?

(iii) what is the amount of heat transferred during this time period?

For steel, $\alpha = 3.954 \times 10^{-6}$ m^2/s, $k = 14.9$ W/(m.C), $\rho = 7900$ kg/m^3, $C_p = 477$ J/(kg.C)

Solution.

Data:

$r_0 := 0.005$ m $\quad \alpha := 3.954\cdot10^{-6}$ m^2/s $\quad k := 14.9$ W/(m.C) $\quad C_p := 477$ J/(kg.C) $\quad \rho := 7900$ kg/m^3

$T_i := 450$°C $\quad T_a := 25$°C $\quad h := 6000$ W/(m^2.C) $\quad T_0 := 50$°C *(centre temp.)*

To calculate: the time τ, temp. at the surface, and amount of heat transferred during this period.

First check if lumped analysis is applicable:

$$Bi := \frac{h\cdot\frac{r_0}{3}}{k}$$ *(define Biot number...for a sphere, $L_c = (V/A) = r_0/3$)*

i.e. $Bi = 0.671$ *(Biot number.)*

It is noted that Biot number is > 0.1; so, **lumped system analysis is not applicable.** We will adopt one term approximation solution.

To find the time required for the centre to reach 50°C:

For one term solution, now, remember that Bi is defined as:

$$Bi := \frac{h\cdot r_0}{k}$$ *(define Biot number)*

i.e. $Bi = 2.013$ *(Biot number)*

Fourier number: $Fo = \dfrac{\alpha\cdot\tau}{r_0^2}$

We have:

$$\theta_0 = \frac{T_0 - T_a}{T_i - T_a}$$ *(T_0 = centre temp. T_i = initial temp., T_a = ambient temp.)*

$$\theta_{0\,sph}(Bi, Fo) := A_{1\,sph}(Bi)\cdot \exp(-\lambda_{1\,sph}(Bi)^2\cdot Fo) \qquad \textit{(Function to determine centre temp. at sphere)}$$

i.e. $$\frac{T_0 - T_a}{T_i - T_a} = 0.05882$$

i.e. the function $\theta_{0\,sph}$ is equal to 0.05882. Let us calculate the Fourier no. to satisfy this requirement. We use the Solve block of Mathcad:

$$Fo := 0.2 \qquad \textit{(guess number)}$$

Given

$$\theta_{0\,sph}(Bi\cdot Fo) = 0.05882$$
$$\text{Find}(Fo) = 0.78$$

i.e. Fourier number: $Fo := 0.78$

Note: Observe the ease with which above calculation is performed with Mathcad.

i.e. $$\tau := \frac{Fo\cdot r_0^2}{\alpha}$$

i.e. $\tau = 4.932$ s *(time reqd. for the centre temp. to reach 50°C...***Ans.***)*

Temperature at the surface of sphere:

At the surface, $r/r_0 = 1.$

i.e. $r_{by}r_0 := 1$

We have:

$$\theta(r, \tau) = \frac{T(r, \tau) - T_a}{T_i - T_a} \qquad \textit{(r = any radius, } r_0 \textit{ = outer radius of cyl.)}$$

$$\theta_{sph}(Bi, Fo, r_{by}r_0) := \left\| \begin{array}{l} A_{1\,sph}(Bi)\cdot \exp(-\lambda_{1\,sph}(Bi)^2\cdot Fo) \text{ if } r_{by}\,r_0 = 0 \\ A_{1\,sph}(Bi)\cdot \exp(-\lambda_{1\,sph}(Bi)^2\cdot Fo)\cdot \dfrac{\sin(\lambda_{1\,sph}(Bi)\cdot r_{by}\,r_0 = 0}{(\lambda_{1\,sph}(Bi)\cdot r_{by}\,r_0 = 0)} \text{ otherwise} \end{array} \right.$$

(Function to determine temp. at any location)

i.e. $\theta_{sph}(Bi, Fo, l) = 0.026$

And, $T := 0.026\cdot(T_i - T_a) + T_a$

i.e. $T = 36.05°C$ *(temp. at the surface of sphere...***Ans.***)*

Amount of heat transferred, Q:

Now, $Q_{max} = \rho\cdot V\cdot C_p\cdot (T_i - T_a)$ *(max. heat transfer possible)*

i.e. $$Q_{max} := \rho\cdot\left(\frac{4\cdot\pi\cdot r_0^3}{3}\right)\cdot C_p\cdot(T_i - T_a) \qquad \textit{(define } Q_{max}\textit{)}$$

i.e. $Q_{max} = 838.558$ J *(max. heat transfer)*

We have:

$$Q_{by}Q_{max\,sph}(Bi, Fo) := 1 - 3\cdot Q_{0\,sph}(Bi, Fo)\cdot\left(\frac{\sin(\lambda_{1\,sph}(Bi) - \lambda_{1\,sph}(Bi)\cdot\cos(\lambda_{1\,sph}(Bi))}{\lambda_{1\,sph}(Bi)^3}\right)$$

(Function to determine Q/Q_{max})

i.e. $Q_{by}Q_{max\,sph}(Bi, Fo) = 0.962$

Therefore, $Q := Q_{max}\cdot 0.962$

i.e. $Q = 806.693$ J *(amount of heat transferred when the centre of sphere reaches 50°C, i.e. in 4.932 seconds...***Ans.***)*